Applied Mathematics and Computation 224 (2013) 88–95

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Singular perturbations of third-order nonlinear differential equations with full nonlinear boundary conditions ☆

Xiaojie Lin

School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, Jiangsu 221116, China

ARTICLE INFO

Keywords:
Singular perturbations
Full nonlinear problems
Asymptotic estimates
Upper and lower solutions
Leray–Schauder degree

ABSTRACT

In this paper, we discuss singular perturbations of third-order nonlinear ordinary differential equations with full nonlinear boundary conditions. The emphasis here is that the nonlinear term depends on the first, second order derivatives and the boundary conditions are full nonlinear that is where the main novelty of this work lies. By applying the upper and lower solutions method, as well as analysis technique, the existence, uniqueness results for the singularly perturbed boundary value problem are established and asymptotic estimates of solutions is also obtained.

1. Introduction

Singularly perturbed boundary value problems (BVP, for short) arise very frequently in fluid mechanics and other branches of applied mathematics. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts and varies slowly in some other parts. There are many results on the existence and asymptotic estimates of solutions for third order singularly perturbed boundary value problems [1,4–6,8–11] and therein. Many techniques arose in the studies of this kind of problems. For example, Howes [6] has considered problems of type

$$\varepsilon^2 y''' = f(y)y' + g(x,y),$$
$$y(a) = A, \quad y(b) = B, \quad y'(b) = C,$$

and discussed the existence and asymptotic estimates of the solutions by the method of descent. Zhao [10] has discussed a more general class of a third order singularly perturbed boundary value problems of the form

$$\varepsilon y''' = f(x,y,y',\varepsilon),$$
$$y'(0) = 0, \quad y(1) = 0, \quad y'(1) = 0,$$

and discussed the existence of solution and obtained asymptotic estimates using the theory of differential inequalities. Feckon [5] has studied high order problems and his approach was based on the nonlinear analysis involving fixed-point theory, Leray–Schauder theory. Valarmathi and Ramanujam [9] have considered singularly perturbed third-order ordinary differential equations of Convection–Diffusion type by using of an asymptotic numerical method. Du et al. [4] were concerned with the existence, uniqueness and asymptotic estimates of solutions of third order multi-point singularly perturbed boundary value problems were give by employing priori estimates, differential inequalities technique and Leray–Schauder degree theory. Lin [8] studied a two-point singularly perturbed boundary value problem to a class of nonlinear vector third order

☆ This work is supported by the Natural Science Foundation of China (Grant No. 11071205 and 11101349), and partially supported by the Natural Science Foundation of Jiangsu Province (Grant No. BK2011042 and 11KJB110013) and PAPD of Jiangsu Higher Education Institutions.

E-mail address: linxiaojie1973@163.com

0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2013.08.048

integro-differential equation and proved the existence of the perturbed solution, also obtained the uniformly valid asymptotic expansion by employing the method of differential inequalities.

However, the boundary conditions in the above-mentioned singularly perturbed boundary value problems for third-order nonlinear differential equation are all linear. Third order boundary value problems with nonlinear boundary conditions have been studied recently by several authors, see for example the papers [2,3,7]. Du et al. [2,3] have established some existence results for the third order nonlinear scalar and vector full boundary value problem, respectively. Lin et al. [7] discussed the existence and uniqueness results of the following nonlinear multi-point boundary value problem by using the upper and lower solutions method, as well as degree theory

$$x'''(t) + f(t, x(t), x'(t), x''(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$\begin{aligned} &x(0) = 0, \\ &g(x'(0), x''(0), x(\xi_1), x(\xi_2), \ldots, x(\xi_{m-2})) = A, \\ &h(x'(1), x''(1), x(\eta_1), x(\eta_2), \ldots, x(\eta_{n-2})) = B. \end{aligned} \tag{2}$$

By so far, very few results were established for third order nonlinear singularly perturbed boundary value problems with nonlinear boundary conditions.

Motivated by the above works, the purpose of this article is to study the singular perturbations of the following third-order nonlinear ordinary differential equations

$$\varepsilon x'''(t) + f(t, x(t), x'(t), x''(t), \varepsilon) = 0, \quad 0 \leqslant t \leqslant 1, \ 0 < \varepsilon \ll 1, \tag{3}$$

with full nonlinear boundary conditions

$$\begin{aligned} &x(0, \varepsilon) = 0, \\ &g(x'(0, \varepsilon), x''(0, \varepsilon), x(\xi_1, \varepsilon), x(\xi_2, \varepsilon), \ldots, x(\xi_{m-2}, \varepsilon)) = A, \\ &h(x'(1, \varepsilon), x''(1, \varepsilon), x(\eta_1, \varepsilon), x(\eta_2, \varepsilon), \ldots, x(\eta_{n-2}, \varepsilon)) = B, \end{aligned} \tag{4}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and $0 < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < 1, A, B \in R, f : [0,1] \times R^3 \to R$ is continuous, $g : R^m \to R, h : R^n \to R$ are continuous.

In this paper, we discuss the existence, uniqueness and asymptotic estimates of solutions for the singularly perturbed boundary value problem (3) and (4). The emphases here is to construct appropriate upper and lower solutions BVP (3) and (4) not to assume that BVP (3) and (4) exist upper and lower solutions.

The remaining part of this paper is organized as follows: in Section 2, we briefly present some definitions and lemmas which are important to obtain our main results. In Section 3, we obtain the existence and asymptotic estimates of solutions of singularly perturbed BVP (3) and (4) by constructing appropriate upper and lower solutions. We also establish the uniqueness result of BVP (3) and (4).

2. Preliminaries

Definition 1 [7]. Let $\tau : C[0,1] \to R^{m-2}, \rho : C[0,1] \to R^{n-2}$ be defined by

$$\tau z = (z(\xi_1), \ldots, z(\xi_{m-2})), \quad \rho z = (z(\eta_1), \ldots, z(\eta_{n-2})). \tag{5}$$

Particularly, for constant M, we denote

$$\tau M = (M, M, \ldots, M) \in R^{m-2}, \quad \rho M = (M, M, \ldots, M) \in R^{n-2}.$$

Definition 2 [7]. A function $u(t) \in C^3[0,1]$ is called a lower solution of BVP (1) and (2), if

$$u'''(t) + f(t, u(t), u'(t), u''(t)) \geqslant 0, \quad 0 \leqslant t \leqslant 1, \tag{6}$$

and

$$\begin{aligned} &u(0) = 0, \\ &g(u'(0), u''(0), \tau u) \leqslant A, \\ &h(u'(1), u''(1), \rho u) \leqslant B. \end{aligned} \tag{7}$$

Similarly, a function $v(t) \in C^3[0,1]$ is called an upper solution of BVP (1) and (2), if

$$v'''(t) + f(t, v(t), v'(t), v''(t)) \leqslant 0, \quad 0 \leqslant t \leqslant 1, \tag{8}$$

and