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The steepest descent algorithm without line search for p-Laplacian

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ABSTRACT

In this paper, the steepest descent algorithm without line search is proposed for p-Laplacian. Its search direction is the weighted preconditioned steepest descent one, and step length is estimated by a formula except the first iteration. Continuation method is applied for solving the p-Laplacian with very large p. Lots of numerical experiments are carried out on these algorithms. All numerical results show the algorithm without line search can cut down some computational time. Fast convergence of these new algorithms is displayed by their step length figures. These figures show that if search direction is the steepest descent one, exact step lengths can be substituted properly with step lengths obtained by the formula.

1. Introduction

The form of the p-Laplacian with Dirichlet data is as follows

$$-div(|\nabla u|^{p-2}\nabla u) = f, \quad \text{in } \Omega, u = 0, \quad \text{on } \partial\Omega, \tag{1}$$

where Ω is a bounded open subset of R^2 with a Lipschitz boundary $\partial\Omega, 1 < p < +\infty, f \in L^2(\Omega)$, and $|\cdot|^2 = (\cdot,\cdot)_{R^2}$. It is well known that the Eq. (1) is Poisson equation when $p = 2$. The Eq. (1) is an important nonlinear equation. It comes from many physical problems, for instance, nonlinear diffusion and filtration (See [1]), power-law materials (see [2]), and quasi-Newtonian flows (see [3]). There are lots of literature on finite element approximation of the p-Laplacian. One can see the literatures [1–11] and the references therein. Of course, some results concerning sharp a priori and a posteriori error bounds of the equation owe to quasi-norm approach (see [12–14]).

The problem (1) is equivalent to the following minimization problem:

$$\min_{v\in V} J(v), \tag{2}$$

where

$$J(v) = \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega f v, \tag{3}$$

and V is the Sobolev space $W_0^{1,p}(\Omega)$. The space is the closure of the space $C_0^\infty(\Omega)$ in the space $W^{1,p}(\Omega)$. Given a two-index $\alpha = (\alpha_1, \alpha_2) \in N^2$, we let $|\alpha| = \sum_{i=1}^2 \alpha_i$. When equipped with the norm

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$$\|v\|_{1,p,\Omega} = (\sum_{|\alpha|\leq 1}\int_\Omega |\partial^\alpha v|^p dx)^{\frac{1}{p}},$$

the space $W^{1,p}(\Omega)$ is a Banach space which, for any number p satisfying $1 < p < +\infty$, consists of those functions $v \in L^p(\Omega)$ for which all partial derivatives $\partial^\alpha v$ with $|\alpha| \leqslant 1$ belong to the space $L^p(\Omega)$. One can refer many references, for example, [15], for the details of these symbols $W_0^{1,p}(\Omega), \partial^\alpha v$ etc.

A direct calculation yields that

$$J'(v)(\tilde{v}) = \int_\Omega |\nabla v|^{p-2}\nabla v \nabla \tilde{v} - \int_\Omega f\tilde{v}, \quad \forall\ \tilde{v} \in V. \tag{4}$$

The problem (2) has a unique solution which is also the solution of the system (1) (See [15]). The form of the Eq. (1) is simple. But how to solve it numerically is a very difficult problem when p is very big ($p \gg 2$) and p is close to 1^+. The status is improved primely since the steepest descent algorithm [16] is proposed. The algorithm can effectively solve the equivalent problem of the p-Laplacian with large p, and has some good properties. Its basic thought will be given in next section. Based on the weighted preconditioned steepest descent direction, preconditioned hybrid conjugate gradient algorithm for the p-Laplacian is proposed [17], and it has higher efficiency than the algorithm [16] when p is large. In order to solve the problem (2), we introduce the following finite element space like [15]. Let T^h be a regular triangulation of Ω_h, which is composed of disjoint open regular triangles K_i, that is, $\bar{\Omega}_h = \bigcup_{K_i\in T^h}\bar{K}_i$, where $h = \max_{K\in T^h} h_K$, and h_K is the diameter of the element K in T^h. $\bar{K}_i \bigcap \bar{K}_j$ is void, or only one common vertex, or a whole edge when $i \neq j$. In this paper, we shall only discuss the continuous piecewise linear element. Let V^h be a finite dimensional subspace of $C^0(\bar{\Omega}_h)$, in which $\chi|_K \in \mathcal{P}_1$ if $\chi \in V^h$ and $\mathcal{P}_1$ is the linear function space. Denote

$$V_0^h = \{\chi \in V^h : \chi(x^k) = 0, \quad \text{for all}\, x^k \in \partial\Omega_h\}. \tag{5}$$

Then the finite element approximation of (2) is as follows:

$$\min_{v_h \in V_0^h} J(v_h), \tag{6}$$

where

$$J(v_h) = \frac{1}{p}\int_{\Omega_h} |\nabla v_h|^p - \int_{\Omega_h} f v_h. \tag{7}$$

The problem (6) has a unique solution which is an approximative solution of the system (1).

General iterative formula solving the unconstrained optimization such as the problem (6) is as follows

$$u_{n+1} = u_n + \alpha_n d_n, \tag{8}$$

where $u_n \in V_0^h$ is the current approximative solution, and α_n is step length on search direction d_n. Search direction d_n is diversified, for example, the steepest descent one (see [16]), hybrid conjugate gradient ones (see [17]) and so on. α_n is usually decided by some line searches, including exact and inexact searches. It is well known that a line search procedure leads to many computations which are dedicated to obtain objective function values and gradients. In order to increase efficiencies of these algorithms, line search procedure in algorithm design should be avoided to the best of one's ability. Based on the thought, for the following unconstrained problem:

$$\min_{x\in R^n} f(x), \tag{9}$$

many algorithms are presented. Dixon [18] proposed a conjugate direction method without line search. Barzilai and Borwein [19] presented a gradient method without line search which is called BB method. Its step length has certain quasi-Newton property. Since the method often requires less computational work and it is efficient on numerical testing, it has now gotten a great deal of attention in optimization community. The attention includes step length designing, convergent analysis, numerical experiments and applications etc. One can refer to [20–27] and the references therein for details. Convergent properties of some conjugate gradient methods without line search were investigated in [18,28,29]. Based on supervisor and searcher cooperation, Liu and Dai [30] presented an algorithm without line search. Numerical effect of the algorithm is good. Vrahatis et al. [27] proposed a class of gradient algorithm with adaptive stepsize for the problem Eq. (9) and Shi [31] proved its convergence. Especially, Shi and Shen [32] proposed a new descent method without line search.

In [16], the problem how to reduce computational time consuming on line search is discussed. The authors proved convergence of a descent algorithm with fixed step length solving the p-Laplacian. But the fixed step length depends on two parameters M and σ. From the proof of Theorem 2 in [16] we can see that the two parameters are difficult to define in real computing. In fact, no numerical experiment is implemented on algorithm with the fixed step length in [16]. In this paper, in order to increase efficiency of algorithm solving the problem Eq. (6), we will propose some algorithms without line search.

The rest of this paper is organized as follows. In Section 2, some preliminaries are given. In Section 3, some algorithms without line search for the p-Laplacian are proposed. Their numerical experiments are performed in Section 4. For dealing with the p-Laplacian with large p, two corresponding algorithms are presented, in which continuation method is used, and