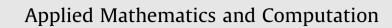
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A solution to an open problem on the Euler–Mascheroni constant

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ARTICLE INFO

Keywords: Euler–Mascheroni constant Bernoulli numbers Le Verrier method Harmonic number ABSTRACT

The classical Le Verrier algorithm for the calculation of the characteristic polynomial of a matrix has led us to a solution of an open problem concerning the Euler–Mascheroni constant γ . More exactly, we prove that there exists uniquely a sequence of rational numbers $(\ell_n)_{n \ge 1}$ such that, for any integer $q \ge 1$, the sequence

$$\sum_{k=1}^n \frac{1}{k} - \log n - \log \left(1 + \frac{\ell_1}{n} + \dots + \frac{\ell_q}{n^q} \right),$$

is an n^{-q-1} order approximation of γ .

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1. Introduction

Let us consider the celebrated Euler sequence

$$\gamma_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n, \quad n \ge 1.$$

The limit of this sequence, the Euler–Mascheroni constant γ , is one of the most important constants in calculus and number theory. There has been a significant amount of interest and research on γ as testified by survey papers (cf., [4]) and expository books (cf., [8]), which reveal its essential properties and surprising connections with other areas of mathematics. However, the sequence (γ_n) converges slowly, and finding alternative sequences with an increased convergence speed has been of major interest.

In 1993, DeTemple [5] introduced the following approximation of order n^{-2} of γ

$$\gamma_n - \log\left(1 + \frac{1}{2n}\right). \tag{1}$$

In 2004, Ivan [9], P 4.1, by a simple application of the Stolz-Cesaro rule, deduced another approximation of order n^{-2} of γ ,

$$\gamma_n - \frac{1}{2} \log \left(1 + \frac{1}{n} \right)$$

He also mentioned that by applying repetitively the Stolz–Cesaro rule, one can obtain asymptotic expansions of any order for γ_n .

In 1997, Negoi [14] deduced an approximation of order n^{-3} ,

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0096-3003/\$ - see front matter © 2013 Elsevier Inc. All rights reserved. http://dx.doi.org/10.1016/j.amc.2013.08.046





$$\gamma_n - \log\left(1 + \frac{1}{2n} + \frac{1}{24n^2}\right). \tag{2}$$

In 2011, Chen and Mortici [2] obtained an approximation of order n^{-5} ,

$$\gamma_n - \log\left(1 + \frac{1}{2n} + \frac{1}{24n^2} - \frac{1}{48n^3} + \frac{23}{5760n^4}\right),\tag{3}$$

and proposed the following open problem.

Open Problem 1 [2]. For a given positive integer q, find the constants a_i (i = 1, ..., q) such that

$$\gamma_n - \log\left(1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \dots + \frac{a_q}{n^q}\right),$$

is the fastest sequence which would converge to γ .

Euler–Mascheroni-type sequences with an improved order of convergence were also considered by Chen [1], Mortici [10– 13] and Sintămărian [15–17].

It is worth mentioning also an interesting recent method of Chlebus [3].

2. A solution to the Open Problem

We begin by defining recurrently a sequence $(\ell_n)_{n\geq 1}$ by the following equations:

$$\begin{cases} c_1 &= -\mathbf{B}_1 \\ 2c_2 &= \mathbf{B}_2 + c_1 \mathbf{B}_1 \\ \vdots &= \vdots \\ qc_q &= (-1)^q \mathbf{B}_q - c_1 (-1)^{q-1} \mathbf{B}_{q-1} - c_2 (-1)^{q-2} \mathbf{B}_{q-2} - \dots + c_{q-1} \mathbf{B}_1 \\ \vdots &= \vdots \end{cases}$$

 $\ell_n := (-1)^{n+1} c_n, \quad n = 1, 2, \dots,$

where \mathbf{B}_n denotes the *n*th Bernoulli number. The sequence $(\ell_n)_{n \ge 1}$ will be crucial in solving the open problem.

Let $q \ge 1$ be an integer.

The following theorem is an answer to the Open Problem 1.

Theorem 2. A sequence of the form

$$\gamma_n - \log\left(1 + \frac{a_1}{n} + \dots + \frac{a_q}{n^q}\right)$$

is an n^{-q-1} order approximation of γ if and only if

$$a_i = \ell_i, \quad i = 1, 2, \ldots, q.$$

Proof. We are looking for real numbers a_1, \ldots, a_q so that the following statement holds true,

$$\gamma_n - \log\left(1 + \frac{a_1}{n} + \dots + \frac{a_q}{n^q}\right) = \gamma + o(n^{-q}) \quad (n \to \infty).$$
(5)

It is obvious that for any real numbers a_1, \ldots, a_q , there exists uniquely a set of complex numbers $\{x_1, \ldots, x_q\}$ such that

$$1 + \frac{a_1}{n} + \dots + \frac{a_q}{n^q} = \left(1 + \frac{x_1}{n}\right) \dots \left(1 + \frac{x_q}{n}\right) \tag{6}$$

We will use the following Newton-Mercator expansion of the principal branch of the complex logarithm,

$$\log\left(1+\frac{z}{n}\right) = \sum_{i=1}^{q} \frac{(-1)^{i-1} z^{i}}{i n^{i}} + o(n^{-q}) \quad (|z| < 1, \ n \to \infty).$$
⁽⁷⁾

We will also use the following well-known asymptotic expansion of the Euler sequence (see, e.g., [8,7], Section 8.367, Entry 13),

$$\gamma_n - \gamma = -\sum_{i=1}^q \frac{\mathbf{B}_i}{i} \frac{1}{n^i} + o(n^{-q}) \quad (n \to \infty).$$
(8)

Recall that if a sequence has an asymptotic expansion, it is unique. This will lead to the uniqueness of the sequence $(a_q)_{q \ge 1}$.

(4)

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