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A solution to an open problem on the Euler–Mascheroni constant

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ABSTRACT

The classical Le Verrier algorithm for the calculation of the characteristic polynomial of a matrix has led us to a solution of an open problem concerning the Euler–Mascheroni constant γ. More exactly, we prove that there exists uniquely a sequence of rational numbers $(\ell_n)_{n\geqslant 1}$ such that, for any integer $q \geqslant 1$, the sequence

$$\sum_{k=1}^{n}\frac{1}{k} - \log n - \log\left(1+\frac{\ell_1}{n}+\cdots+\frac{\ell_q}{n^q}\right),$$

is an n^{-q-1} order approximation of γ.

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1. Introduction

Let us consider the celebrated Euler sequence

$$\gamma_n = 1+\frac{1}{2}+\cdots+\frac{1}{n} - \log n, \quad n \geqslant 1.$$

The limit of this sequence, the Euler–Mascheroni constant γ, is one of the most important constants in calculus and number theory. There has been a significant amount of interest and research on γ as testified by survey papers (cf., [4]) and expository books (cf., [8]), which reveal its essential properties and surprising connections with other areas of mathematics. However, the sequence (γ_n) converges slowly, and finding alternative sequences with an increased convergence speed has been of major interest.

In 1993, DeTemple [5] introduced the following approximation of order n^{-2} of γ

$$\gamma_n - \log\left(1+\frac{1}{2n}\right). \tag{1}$$

In 2004, Ivan [9], P 4.1, by a simple application of the Stolz-Cesaro rule, deduced another approximation of order n^{-2} of γ,

$$\gamma_n - \frac{1}{2}\log\left(1+\frac{1}{n}\right).$$

He also mentioned that by applying repetitively the Stolz–Cesaro rule, one can obtain asymptotic expansions of any order for γ_n.

In 1997, Negoi [14] deduced an approximation of order n^{-3},

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$$\gamma_n - \log\left(1 + \frac{1}{2n} + \frac{1}{24n^2}\right). \tag{2}$$

In 2011, Chen and Mortici [2] obtained an approximation of order n^{-5},

$$\gamma_n - \log\left(1 + \frac{1}{2n} + \frac{1}{24n^2} - \frac{1}{48n^3} + \frac{23}{5760n^4}\right), \tag{3}$$

and proposed the following open problem.

Open Problem 1 [2]. *For a given positive integer q, find the constants a_i $(i = 1, \ldots, q)$ such that*

$$\gamma_n - \log\left(1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots + \frac{a_q}{n^q}\right),$$

is the fastest sequence which would converge to γ.

Euler–Mascheroni-type sequences with an improved order of convergence were also considered by Chen [1], Mortici [10–13] and Sîntămărian [15–17].

It is worth mentioning also an interesting recent method of Chlebus [3].

2. A solution to the Open Problem

We begin by defining recurrently a sequence $(\ell_n)_{n\geqslant 1}$ by the following equations:

$$\begin{cases} c_1 &= -\mathbf{B}_1 \\ 2c_2 &= \mathbf{B}_2 + c_1\mathbf{B}_1 \\ \vdots &= \vdots \\ qc_q &= (-1)^q\mathbf{B}_q - c_1(-1)^{q-1}\mathbf{B}_{q-1} - c_2(-1)^{q-2}\mathbf{B}_{q-2} - \cdots + c_{q-1}\mathbf{B}_1, \\ \vdots &= \vdots \end{cases}$$

$$\ell_n := (-1)^{n+1}c_n, \quad n = 1, 2, \ldots, \tag{4}$$

where $\mathbf{B}_n$ denotes the nth Bernoulli number. The sequence $(\ell_n)_{n\geqslant 1}$ will be crucial in solving the open problem.

Let $q \geqslant 1$ be an integer.

The following theorem is an answer to the Open Problem 1.

Theorem 2. *A sequence of the form*

$$\gamma_n - \log\left(1 + \frac{a_1}{n} + \cdots + \frac{a_q}{n^q}\right)$$

is an n^{-q-1} order approximation of γ if and only if

$$a_i = \ell_i, \quad i = 1, 2, \ldots, q.$$

Proof. We are looking for real numbers $a_1, \ldots, a_q$ so that the following statement holds true,

$$\gamma_n - \log\left(1 + \frac{a_1}{n} + \cdots + \frac{a_q}{n^q}\right) = \gamma + o(n^{-q}) \quad (n \to \infty). \tag{5}$$

It is obvious that for any real numbers $a_1, \ldots, a_q$, there exists uniquely a set of complex numbers $\{x_1, \ldots, x_q\}$ such that

$$1 + \frac{a_1}{n} + \cdots + \frac{a_q}{n^q} = \left(1 + \frac{x_1}{n}\right) \ldots \left(1 + \frac{x_q}{n}\right) \tag{6}$$

We will use the following Newton-Mercator expansion of the principal branch of the complex logarithm,

$$\log\left(1 + \frac{z}{n}\right) = \sum_{i=1}^{q} \frac{(-1)^{i-1}z^i}{in^i} + o(n^{-q}) \quad (|z| < 1,\ n \to \infty). \tag{7}$$

We will also use the following well-known asymptotic expansion of the Euler sequence (see, e.g., [8,7], Section 8.367, Entry 13),

$$\gamma_n - \gamma = -\sum_{i=1}^{q} \frac{\mathbf{B}_i}{i} \frac{1}{n^i} + o(n^{-q}) \quad (n \to \infty). \tag{8}$$

Recall that if a sequence has an asymptotic expansion, it is unique. This will lead to the uniqueness of the sequence $(a_q)_{q\geqslant 1}$.

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