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Complexity of solving nonlinear equations in the deterministic, randomized and quantum settings ☆

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ABSTRACT

We consider the root finding of a real-valued function f defined on the d-dimensional unit cube. We assume that f has r continuous partial derivatives, with all partial derivatives of order r being Hölder functions with the exponent ρ. We study the ε-complexity of this problem in three settings: deterministic, randomized and quantum. It is known that with the root error criterion the deterministic ε-complexity is infinite, i.e., the problem is unsolvable. We show that the same holds in the randomized and quantum settings. Under the residual error criterion, we show that the deterministic and randomized ε-complexity is of order $\varepsilon^{-d/(r+\rho)}$. In the quantum setting, the ε-complexity is shown to be of order $\varepsilon^{-d/(2(r+\rho))}$. This means that a quadratic speed-up is achieved on a quantum computer.

1. Introduction

For many computational problems the complexity (the minimal cost of solving the problem) strongly depends on the setting under consideration. Traditionally, the most used setting has been the deterministic one. For many years, randomized algorithms have also been investigated, and in recent years algorithms designed for a potential quantum computer are widely studied. It turns out that quantum algorithms often yield a speed-up over the deterministic or randomized ones. Many discrete problems have been studied from that point of view, to mention important contributions such as Shor's factorization algorithm [17] and Grover's database search algorithm [5], or algorithms for computing discrete statistics such as the mean, median, k-th smallest element (see e.g. [3,6,12]).

On the other hand, many continuous (numerical) problems have also been investigated in the deterministic, randomized and quantum settings. Within the latter setting for continuous problems, the first work concerning the efficiency of quantum computations was the paper of Novak [14] devoted to multivariate integration. After that the approximation problem was investigated in [8,9], and ordinary differential equations in [10]. For the maximization of a function from a Hölder class, a quadratic speed-up of quantum computation in comparison to the deterministic and randomized computation was shown in [4].

In this paper, we study the problem of finding the root of a function from a Hölder class of functions defined on the d-dimensional unite cube. Three settings are considered: the deterministic, randomized and quantum ones. The most natural error criterion for root finding is defined by the distance between the exact and approximate solution (the root criterion). It turns out that the root criterion is very much demanding, and it leads to negative results. Hence, along with that we shall also study the residual error criterion.

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In the literature, the root criterion in the deterministic worst-case setting was studied in [19]. The authors showed that it was not possible to find the solution satisfying the root criterion with precision $0 < \varepsilon < 1/2$, even for arbitrary regular functions and if the computation of any adaptive linear information was allowed. The residual criterion in the deterministic worst-case setting in the Lipschitz class of functions was investigated by Sikorski [18]. He established upper and lower ε-complexity bounds for this problem of order ε^{-d} and constructed an optimal algorithm.

In the present paper, we consider the Hölder class of functions consisting of r times differentiable functions with partial derivatives of order r satisfying the Hölder condition with the exponent ρ. We first consider the root criterion in the randomized and quantum settings. We show that allowing randomized algorithms or even turning to quantum computations does not help us to solve the problem: the ε-complexity remains infinite, as in the deterministic setting. Thus, the problem is unsolvable under the root error criterion.

In the light of that, we turn to the residual error criterion. We first construct algorithms for solving the problem in the deterministic worst-case, randomized and quantum settings. The algorithms are based on the optimal discrete search algorithms, in particular Grover's algorithm in the quantum setting. The error analysis of these algorithms allows us to establish upper ε-complexity bounds. By considering intrinsic difficulties encountered by an arbitrary algorithm, we also show matching lower bounds on the ε-complexity. As the result, in the deterministic and randomized settings the ε-complexity is shown to be of order $\varepsilon^{-d/(r+\rho)}$, and in the quantum setting of order $\varepsilon^{-d/(2(r+\rho))}$. This means that for the residual error criterion, a quadratic speed-up on a quantum computer is achieved for root finding problem over a classical computer in both the deterministic and randomized settings.

The paper is organized as follows: in Section 2 the problem is formulated. In Section 3, known results about the problem are recalled together with some results useful in further analysis. In Section 4 we show that the randomized and quantum complexities with the root criterion are infinite. Section 5 contains the definition of algorithms and upper complexity bounds with the residual criterion, and Section 6 the matching lower complexity bounds.

2. Problem formulation and basic definitions

Let $f : [0,1]^d \to \mathbf{R}$ be a real-valued function for which there exists a point $t^0 \in [0,1]^d$ such that $f(t^0) = 0$. We are interested in finding approximate solution of the equation $f(t) = 0$ with precision $\varepsilon > 0$ in the sense of the root or the residual error criterion. That is, we wish to compute a point $t^* \in [0,1]^d$ for which $|t^* - t^0| \leqslant \varepsilon$ for some solution t^0 (the root criterion), or $|f(t^*)| \leqslant \varepsilon$ (the residual criterion). We consider a Hölder class of functions f given by

$$F_d^{r,\rho} = \left\{ f : [0,1]^d \to \mathbf{R} \middle| f \in C^r([0,1]^d), |D^{(i)} f(x)| \leqslant D, \quad i = 0,1,\ldots r, |D^{(r)} f(x) - D^{(r)} f(y)| \leqslant H\|x-y\|^\rho \text{ for } x,y \in [0,1]^d \right\},$$

where $r \geqslant 0$, $0 < \rho \leqslant 1$ and D, H are positive constants. Here $D^{(i)}$ runs through the set of all partial derivatives of order i, and $\|\cdot\|$ means the maximum norm on $\mathbf{R}^d$, $\|\cdot\| = \|\cdot\|_\infty$. The class of functions $f \in F_d^{r,\rho}$ having at least one zero $t^0 \in [0,1]^d$ will be denoted by $F_d^{r,\rho}(0)$. We shall analyze the problem of finding root of a function $f \in F_d^{r,\rho}(0)$ in three settings: the deterministic worst-case, the randomized and the quantum settings.

To find the approximation t^* we need some information about the function $f \in F_d^{r,\rho}$. In the deterministic setting, by information we mean an operator N that assigns to the function f a vector $N(f)$ of length at most n, whose components are the values of f or its derivatives of order at most r at some (adaptively chosen) deterministic points from $[0,1]^d$.

Let $(\Omega, \Sigma, \mathrm{P})$ be a probability space. In the randomized setting, by information we mean a family of operators $N = \{N^\omega\}_{\omega\in\Omega}$. Here, $N^\omega(f)$ is defined by the values of f or its derivatives computed at randomly chosen points from $[0,1]^d$ (determined by the choice of ω). We assume that the points x_i and derivative numbers k_i in evaluation of the successive $f^{(k_i)}(x_i)$ are chosen adaptively as functions of information values computed so far. The decision whether to compute a successive piece of information is also taken based on the computed values, according to some termination criterion. We assume that the length $n^\omega(f)$ of the information vector is a measurable function and it satisfies $\mathrm{E}(n^\omega(f)) \leqslant n$ for all $f \in F_d^{r,\rho}$.

In the quantum setting, information is gathered by applying n times a (standard) quantum query operator about the function f. For a detailed description of the quantum model of computation, quantum algorithms, and the quantum query operator, the reader is referred to [7,11]. Roughly speaking, a quantum query is a unitary operator defined by f whose application plays a role of evaluating the value of a function or its derivative in the standard deterministic setting.

The approximate solution t^* is obtained in the deterministic setting by an algorithm ϕ that computes t^* based on the information values $N(f)$, $t^* = \phi(N(f))$. In the randomized setting, the algorithm is a family $\phi = \{\phi^\omega\}_{\omega\in\Omega}$, and $t^* = t^{*(\omega)} = \phi^\omega(N^\omega(f))$. In the quantum setting, the algorithm is defined by an application of a number of unitary operations, n of them being quantum queries, followed by a quantum measurement. Further possible operations performed on a classical computer lead to $t^* = t^{*(\omega)}$, see [7,11]. In the randomized and quantum settings, the point $t^* = t^{*(\omega)}$ is a random vector.

We shall consider the root and residual error criterions. The superscript 'ω' in the definitions below in the deterministic setting is obviously irrelevant, and it will then be omitted.

Let $S(f) = \{x \in [0,1]^d : f(x) = 0\}$. We define the local root error of an algorithm ϕ by

$$\mathrm{e}^\omega_{\mathrm{root}}(f, \phi^\omega) = \mathrm{dist}(t^*, S(f)).$$

The residual error is given by