Applied Mathematics and Computation 224 (2013) 743–759

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Existence of solutions for neutral integro-differential equations with state-dependent delay ☆

Xianlong Fu *, Rong Huang

Department of Mathematics, East China Normal University, Shanghai 200241, PR China

ARTICLE INFO

Keywords:
Neutral integro-differential equations
Analytic semigroup
Resolvent operator
Fractional power operator
State-dependent delay

ABSTRACT

In this paper, we study the existence and regularity of solutions for a neutral functional integro-differential equation with state-dependent delay in Banach space. The mild solutions are investigated by Sadovskii fixed point theorem under compactness condition for the resolvent operator, and the theory of fractional power and α-norm are also used in the discussion since the nonlinear terms of the system involve spacial derivatives. The strict solutions are obtained under Hölder continuity condition. Hence this work extends and develops the existed results on this topic. In the end an example is provided to illustrate the application of the obtained results.

1. Introduction

In this paper, we study the existence of solutions for semilinear neutral integro-differential equations with state-dependent delay of the following form:

$$\begin{cases} \frac{d}{dt}[x(t)+F(t,x_t)] = -Ax(t) + \int_0^t \Upsilon(t-s)x(s)ds + G(t,x_{\rho(t,x_t)}), \quad t \in [0,T], \\ x_0 = \varphi \in \mathcal{B}_\alpha, \end{cases} \tag{1}$$

where $-A$ is the infinitesimal generator of an analytic semigroup on a Banach space X, $\Upsilon(t)$ is a closed linear operator defined later, F, G and ρ are given continuous functions to be specified below, and $\mathcal{B}_\alpha$ is an abstract phase space endowed with a seminorm $\|\cdot\|_{\mathcal{B}_\alpha}$.

Integro-differential equations can be used to describe a lot of natural phenomena arising from many fields such as electronics, fluid dynamics, biological models and chemical kinetics. Most of these phenomena cannot be described through classical differential equations. That is why in recent years they have attracted more and more attention of many mathematicians, physicists, and engineers. Some topics for this kind of equations, such as existence and regularity, stability, (almost) periodicity of solutions and control problems, have been investigated by many mathematicians, see [1–7] for example.

In [8–10], Grimmer et al. proved the existence of solutions of the following integrodifferential evolution equation:

$$\begin{cases} v'(t) = Av(t) + \int_0^t \Upsilon(t-s)v(s)ds + g(t), \quad \text{for } t \geqslant 0, \\ v(0) = v_0 \in X, \end{cases} \tag{2}$$

where $g : \mathbb{R}^+ \to X$ is a continuous function. They obtained the representation of solutions, the existence and uniqueness of solutions via resolvent operator associated to the following linear homogeneous equation

☆ This work is supported by NSF of China (Nos. 11171110, 11371087) and Shanghai Leading Academic Discipline Project (No. B407).
* Corresponding author.
E-mail address: xlfu@math.ecnu.edu.cn (X. Fu).

0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2013.09.010

$$\begin{cases} v'(t) = Av(t) + \int_0^t \Upsilon(t-s)v(s)ds, & \text{for } t \geqslant 0, \\ v(0) = v_0 \in X. \end{cases}$$

That is, the resolvent operator $R(t)$, replacing the role of C_0-semigroup for evolution equations, plays an important role in solving Eq. (2) in weak and strict senses. In recent years much work on existence problems for nonlinear integro-differential evolution equations has been done by many authors through applying the theory of resolvent operator. In paper [11], the authors have discussed the local existence and regularity of solutions for the following partial functional differential equations with infinite delay in Banach space:

$$\begin{cases} u'(t) = Au(t) + \int_0^t \Upsilon(t-s)u(s)ds + f(u_t), & \text{for } t \geqslant 0, \\ u_0 = \varphi \in \mathcal{B}. \end{cases} \tag{3}$$

Under the assumptions that $f : \mathcal{B} \to X$ is continuously differentiable and $f' : \mathcal{B} \to \mathcal{L}(\mathcal{B}, X)$ is locally Lipschitz continuous, they have showed the existence of strict solutions.

Partial neutral integro-differential equations arise, for instance, in the theory of heat conduction in fading memory material. In the classic theory of heat conduction, it is assumed that the internal energy and the heat flux depend linearly on the temperature u and on its gradient ∇u. Under these conditions, the classical heat equation describes very well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [12–15], the internal energy and the heat flux are described as functionals of u and u_x. The next system, see [16–19], has been frequently used to describe this phenomena,

$$\frac{\partial}{\partial t}\left[u(t,x) + \int_{-\infty}^t k_1(t-s)u(s,x)ds\right] = d\triangle u(t) + \int_{-\infty}^t k_2(t-s)\triangle u(s,x)ds, +f(t,u(\cdot,x)), \quad \text{for } t \geqslant 0,\ u(t,x) = 0,\ x \in \partial\Omega.$$

In this system, $\Omega \subset \mathbb{R}^n$ is open, bounded and has smooth boundary, $(t,x) \in [0,\infty) \times \Omega$, $u(t,x)$ represents the temperature in x at time t, d is a physical constant and $k_i : \mathbb{R} \to \mathbb{R}$ $(i = 1,2)$ are the internal energy and the heat flux relaxation respectively. By assuming the solution u is know on $(-\infty, 0]$ we can transform this system into the abstract form (1).

In the past years the nonlinear neutral functional integro-differential equations with resolvent operators has become an active area of investigation, see [20–24] and the references therein. Ezzinbi et al. [21,22] have studied the existence of mild solutions for this type of neutral equations by using Banach fixed point theorem, and the regularity of solutions was discussed there under the conditions that the nonlinear functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are continuously differentiable. In addition, functional differential equations with state-dependent delay appear frequently in various models and hence the study of this kind of equations has also received great attention in the last years. Some recent work can be found in [25–27]. We would like to mention here the related work [28–30] on the topic of existence problems for neutral differential equations for our references.

We are to discuss in this paper the existence of solutions for Eq. (1) with state-dependent delay to extend the work of [21,22,11]. One of our purpose here is to make the established results applied to the following system:

$$\begin{cases} \frac{\partial}{\partial t}\left[z(t,x) + F(t, z(\cdot,x), \frac{\partial}{\partial x}z(\cdot,x))\right] = \frac{\partial^2}{\partial x^2}z(\cdot,x) + \int_0^t \Upsilon(t-s)z(s,x)ds + G\left(t, \frac{\partial z}{\partial x}(\cdot,x)\right), \\ z(t,0) = z(t,\pi) = 0, \quad t \in [0,T], \\ z(\theta,x) = \varphi(\theta,x), \theta \leqslant 0, \quad 0 \leqslant x \leqslant \pi. \end{cases} \tag{4}$$

Clearly, this system can be treated as the abstract Eq. (1), however, the results established in [11,20–23]. become invalid for this situation, since the functions F and G in (4) involve spatial derivatives. As one will see in Section 5, if take $X = L^2([0,\pi])$, then the history variables of F and G are defined on $\mathcal{C}_{g_{\frac{1}{2}}}$ (induced by $X_{\frac{1}{2}}$) and so the solutions can not be discussed on the whole space X like in the literature [11,20–23]. In this paper, inspired by the work in [31–33], we shall discuss this problem by using the theory of fractional power operators and α-norm, that is, we shall restrict this equation in a Banach space $X_\alpha(\subset X)$ and investigate the existence and regularity of mild solutions for Eq. (1) via $\|\cdot\|_\alpha$. To obtain the existence of mild solutions for system (1), we assume that the analytic semigroup $S(t)_{t\geqslant 0}$ generated by $(A, D(A))$ is compact (for $t > 0$, but the resolvent $R(t)$ may be non-compact) and so that the fixed point principle for condensing maps is applied, this is quite different from the works in [11,21–23]. Note that the compactness of $S(t)_{t\geqslant 0}$ does not implies necessarily that the resolvent operator $R(t)$ is compact, our discussion is also different from that of Chang and Li [20] as well. Additionally, we do not require that $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are continuously differentiable as studying the regularity of mild solutions as in [11,21,22], instead, we only suppose that they are Hölder continuous. Clearly, our obtained results extend and develop the existing results, such as the above mentioned ones.

The whole article is arranged as follows: we firstly introduce some preliminaries about analytic resolvent operators and phase space for state-dependent delay in Section 2. Particularly, to make them to be still valid in our situation, we have restated the axioms of phase space on the space X_α. The existence of mild solutions is discussed in Section 3 by applying fixed point theorem. In Section 4, we establish some sufficient conditions to guarantee the regularity of mild solutions, that is, we obtain the existence of strict solutions by Hölder continuity conditions. Finally, in Section 5, an example is presented to show the applications of the obtained results.