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American option pricing under two stochastic volatility processes

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ABSTRACT

In this paper we consider the pricing of an American call option whose underlying asset dynamics evolve under the influence of two independent stochastic volatility processes as proposed in Christoffersen, Heston and Jacobs (2009) [13]. We consider the associated partial differential equation (PDE) for the option price and its solution. An integral expression for the general solution of the PDE is presented by using Duhamel's principle and this is expressed in terms of the joint transition density function for the driving stochastic processes. For the particular form of the underlying dynamics we are able to solve the Kolmogorov PDE for the joint transition density function by first transforming it to a corresponding system of characteristic PDEs using a combination of Fourier and Laplace transforms. The characteristic PDE system is solved by using the method of characteristics. With the full price representation in place, numerical results are presented by first approximating the early exercise surface with a bivariate log linear function. We perform numerical comparisons with results generated by the method of lines algorithm and note that our approach provides quite good accuracy.

1. Introduction

It has long been recognised that the original option pricing framework of Black and Scholes [5] suffers from the deficiency that the Normal distribution is inadequate to pick up the skewness and kurtosis observed in real financial data. There is a mountain of empirical evidence from Mandelbrot [29] to Platen and Rendek [31] to this effect. Equally important has been the impact on European option pricing when the underlying asset is driven by stochastic volatility processes. In this regard, we refer the reader to Scott [32], Wiggins [36], Hull and White [19], Stein and Stein [34], and Heston [17], who all consider European option pricing where the underlying asset dynamics evolve under the influence of various types of single factor mean reverting stochastic volatility processes.

The standard option pricing framework of Black and Scholes [5] has been premised on a number of restrictive assumptions, one of which is constant volatility of asset returns. The constant volatility assumption is based on the early perception that asset returns are characterized by the Normal distribution.

A variety of papers have also proved that multiple factor stochastic volatility models perform quite well relative to single factor models. One such paper is the most recent work by Christoffersen et al. [13] who show empirically with the aid of principal component analysis that a two factor model offers flexibility in controlling the level and slope of the volatility smirk

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(a situation where in- and out-of-the-money options have higher implied volatilities as compared to Black–Scholes prices). It has also been shown empirically by da Fonseca et al. [15] that multiple factor stochastic volatility models offer more consistent option prices as compared to single factor models. As demonstrated in da Fonseca et al. [15] and Christoffersen et al. [13] the main advantages of a multiple volatility system is that they calibrate short-term and long-term volatility levels better than a single process.

Whilst most of the initial work has focused on European style options, not much has been done on pricing American options under stochastic volatility. Touzi (1999) considers the pricing of American put options written on an underlying asset whose dynamics evolve under the influence of stochastic volatility. Touzi describes the dependence of the early exercise boundary of the American put option on the volatility parameter and proves that such a boundary is a decreasing function of volatility implying that for a fixed underlying asset price, as the volatility increases, the early exercise boundary decreases.

Tzavalis and Wang [35] derive the integral representation of an American call option price when the volatility process evolves according to the square-root process proposed by Heston [17]. They derive the integral expressions, again using optimal stopping theory along the lines of Karatzas [25]. By appealing to the empirical findings by Broadie et al. [6] who suggest that when variance evolves stochastically the early exercise boundary is a log-linear function of both time and instantaneous variance, a Taylor series expansion is applied to the resulting early exercise surface around the long-run variance. The unknown functions resulting from the Taylor series expansion are then approximated by fitting Chebyshev polynomials. Ikonen and Toivanen [20] formulate and solve the linear complementarity problem of the American call option under stochastic volatility using componentwise splitting methods. The resulting subproblems from componentwise splitting are solved by using standard partial differential equation methods.

Chiarella and Ziogas [10] derive the integral representation of the American call option pricing partial differential equation (PDE) when the dynamics of the underlying asset is influenced by a single stochastic volatility process by first applying a Fourier transform to the underlying asset domain and then guessing the general solution of the PDE along similar lines as initially presented in [17] when pricing European call options under stochastic volatility. Adolfsson et al. [2] also derive the integral representation of the American call[1] option under stochastic volatility by formulating the pricing PDE as an inhomogeneous problem and then using Duhamel's principle to represent the corresponding solution in terms of the joint transition density function. The joint density function solves the associated backward Kolmogorov PDE and a systematic approach for solving such a PDE is developed. A combination of Fourier and Laplace transforms is used to transform the homogeneous PDE for the density function to a characteristic PDE. The resulting system is then solved using ideas first presented by Feller [16]. The early exercise boundary is approximated by a log-linear function as proposed in [35]. Instead of using approximating polynomials as in [35,2], Adolfsson et al. [2] derive an explicit characteristic function for the early exercise premium component and then use numerical root finding techniques to find the unknown functions from the log-linear approximation.

Motivated by the superior features of multifactor volatility processes as empirically proved in [15,13], we seek to extend the American option pricing model of Adolfsson et al. [2] to the multifactor stochastic volatility case. We will assume that the underlying asset is driven by two stochastic variance processes of Christoffersen et al. [13] type. Whilst da Fonseca [14,15] treat the two stochastic variance processes to be effective during different periods of the maturity domain, in this work we model the variance processes as independent risk factors influencing the dynamics of the underlying asset. As an example, we aim to devise semi-analytical solutions for American call options under this framework that are computationally tractable.

The PDE for the option price is solved subject to initial and boundary conditions that specify the type of option under consideration. As is well known, the underlying asset of the American call option is bounded above by the early exercise boundary and below by zero. We convert the upper bound of the underlying asset to an unbounded domain by using the approach of Jamshidian [21]. The three stochastic processes; one for the underlying asset and the two variance processes can also be used to derive the corresponding PDE for their joint transition probability density function which satisfies a backward Kolmogorov PDE. Coupled with this and the unbounded PDE for the option price, we derive the general solution for the American option price by using Duhamel's principle. The only unknown terms in the general solution are the transition density function which is the solution of the backward Kolmogorov PDE for the three driving processes and the free boundary, for which an integral equation will be found from the initial and boundary conditions.

In solving the Kolmogorov PDE, we first reduce it to a characteristic PDE by using a combination of Fourier and Laplace transforms. The resulting equation is then solved by the method of characteristics. Once the solution is found, we revert back to the original variables by applying the Fourier and Laplace inversion theorems. With the transition density in place, we can readily obtain the full integral representation of the American option price. As implied by Duhamel's principle, the American option price is the sum of two components namely the European and the early exercise premium components. The European option component can be readily reduced to the [17] form by using similar techniques to those in [2]. In dealing with the early exercise premium component, we extend the idea of Tzavalis and Wang [35] and approximate the early exercise boundary as a bivariate log-linear function. This approximation allows us to reduce the integral dimensions of the early exercise premium by simplifying the integrals with respect to the two variance processes. The reduction of the dimensionality has the net effect of enhancing computational efficiency by reducing the computational time of the early exercise premium component.

[1] Their approach can be generalised to any option payoff function under the same framework.