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On the development of iterative methods for multiple roots

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ABSTRACT

There are very few optimal fourth order methods for solving nonlinear algebraic equations having roots of multiplicity m. Here we compare 4 such methods, two of which require the evaluation of the $(m-1)^{st}$ root. We will show that such computation does not affect the overall cost of the method.

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1. Introduction

There is a vast literature on the solution of nonlinear equations and nonlinear systems, see for example Ostrowski [1], Traub [2], Neta [3] and the recent book by Petković et al. [4] and references therein. Most of the algorithms are for finding a simple root of a nonlinear equation $f(x)=0$, i.e. for a root α we have $f(\alpha)=0$ and $f'(\alpha)\neq 0$. In this paper we are interested in the case that α is a root of multiplicity $m>1$. Clearly, one can use the quotient $f(x)/f'(x)$ which has a simple root where $f(x)$ has a multiple root. Such an idea will not require a knowledge of the multiplicity, but on the other hand will require higher derivatives. For example, Newton's method for the function $F(x)=f(x)/f'(x)$ will be

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)-\frac{f(x_n)f''(x_n)}{f'(x_n)}}. \tag{1}$$

If we define the efficiency index of a method of order, p as

$$I=p^{1/d}, \tag{2}$$

where d is the number of function- (and derivative-) evaluation per step then this method has an efficiency of $2^{1/3}=1.2599$ instead of $\sqrt{2}=1.4142$ for Newton's method for simple roots.

There are very few methods for multiple roots when the multiplicity is known. The first one is due to Schröder [5] and it is also referred to as modified Newton,

$$x_{n+1}=x_n-m\,u_n, \tag{3}$$

where

$$u_n=\frac{f(x_n)}{f'(x_n)}. \tag{4}$$

The method is based on Newton's method for the function $G(x)=\sqrt[m]{f(x)}$ which obviously has a simple root at α, the multiple root with multiplicity m of $f(x)$.

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Another method based on the same G is Laguerre's-like method

$$x_{n+1} = x_n - \frac{\lambda u_n}{1 + sgn(\lambda - m)\sqrt{\left(\frac{\lambda - m}{m}\right)\left[(\lambda - 1) - \lambda u_n \frac{f''(x_n)}{f'(x_n)}\right]}} \tag{5}$$

where $\lambda\ (\neq 0, m)$ is a real parameter. When $f(x)$ is a polynomial of degree n, this method with $\lambda = n$ is the ordinary Laguerre method for multiple roots, see Bodewig [6] and Neta and Chun [7]. This family of merthods converges cubically.

We now list optimal fourth order methods for multiple roots. The first paper is by Li et al. [8]. Their method is a special case of one of the families found later by Li et al. [9].

Li et al. [9] have developed six fourth order methods based on the results of Neta and Johnson [10] and Neta [11]. Here we list the two optimal fourth order.

- LCN5

$$\begin{aligned} y_n &= x_n - \frac{2m}{m+2}u_n, \\ x_{n+1} &= x_n - a_3\frac{f(x_n)}{f'(y_n)} - \frac{f(x_n)}{b_1 f'(x_n) + b_2 f'(y_n)}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} a_3 &= -\frac{1}{2}\frac{\left(\frac{m}{m+2}\right)^m m(m-2)(m+2)^3}{m^3-4m+8}, \\ b_1 &= -\frac{(m^3-4m+8)^2}{m(m^4+4m^3-4m^2-16m+16)(m^2+2m-4)}, \\ b_2 &= \frac{m^2(m^3-4m+8)}{\left(\frac{m}{m+2}\right)^m (m^4+4m^3-4m^2-16m+16)(m^2+2m-4)}. \end{aligned}$$

- LCN6

$$\begin{aligned} y_n &= x_n - \frac{2m}{m+2}u_n, \\ x_{n+1} &= x_n - a_3\frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)}{b_1 f'(x_n) + b_2 f'(y_n)}, \end{aligned} \tag{7}$$

where

$$a_3 = -\frac{1}{2}m(m-2),$$

$$b_1 = -\frac{1}{m}, \quad b_2 = \frac{1}{m\left(\frac{m}{m+2}\right)^m}.$$

Zhou et al. [14] have also developed fourth-order optimal methods for multiple roots but they will not be included in the comparison given here. We now give the optimal methods due to Liu and Zhou [12]. These methods require the computation of the $\sqrt[(m-1)]{\frac{f'(y_n)}{f'(x_n)}}$.

Two methods from the family developed by Liu and Zhou [12]

$$\begin{aligned} y_n &= x_n - mu_n, \\ x_{n+1} &= x_n - mH(w_n)\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{8}$$

where

$$w_n = \sqrt[(m-1)]{\frac{f'(y_n)}{f'(x_n)}} \tag{9}$$

and $H(0) = 0, \quad H'(0) = 1, \quad H''(0) = \frac{4m}{m-1}$.

The two members given there are.

- LZ11