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On solutions of the second generalization of d'Alembert's functional equation on a restricted domain

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ABSTRACT

Let A be a subgroup of an abelian group $(G,+)$ and P be a quadratically closed field with char $P \neq 2$. We give a full description of all pairs of functions $f : G \to P,\ g : A \to P$ satisfying the equation

$$f(x+y)+f(x-y)=2g(x)f(y) \quad (x,y)\in A\times G. \tag{a}$$

We present an example of solution (f,g) of (a) that cannot be extended to a solution $(f,\overline{g})$ of the equation

$$f(x+y)+f(x-y)=2\overline{g}(x)f(y) \quad x,y\in G. \tag{b}$$

1. Introduction

Many authors have investigated the solutions of Cauchy functional equation on a restricted domain (see for example [6,7,9,11,12,16]. In particular the papers [6, Théorème 1] and [7, Theorems 1 and 2]) have proved the following

Theorem 1.1. *Let $(H,\cdot)$ and $(F,\cdot)$ be groups, $\emptyset \neq Y \subset H$ and $Z = H\times Y$. Assume that F is of order greater than 2. Then the following two statements are valid.*

(i) A solution $h : H \to F$ of

$$h(xy)=h(x)h(y) \quad (x,y)\in Z, \tag{1.1}$$

is a homomorphism if and only if the subgroup generated by Y is H.

(ii) If H is abelian, then $h : H \to F$ satisfies (1.1) if and only if

$$h(x)=g(x\xi(\pi(x))^{-1})\lambda(\pi(x)) \quad x\in H,$$

for some $\lambda : H/H_0 \to F$ with $\lambda(e)=e$ and some homomorphism $g : H_0 \to F$, where H_0 is the subgroup of H that is generated by Y, e denotes the neutral elements in H/H_0 and F, $\pi : H \to H/H_0$ is the natural projection and $\xi : H/H_0 \to H$ is a lifting (i.e., $\pi(\xi(u))=u$ for $u\in H/H_0$).

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In the paper [5] we obtained similar results for the d'Alembert functional equation

$$h(x+y)+h(x-y)=2h(x)h(y), \tag{1.2}$$

in the class of functions mapping an abelian group $(G,+)$ into an abelian ring $(P,+,\cdot)$. In that paper the study was restricted to the situation in which (1.2) is valid for every $(x,y)\in A\times G$ with A being a subgroup of G.

The d'Alembert functional equation and its generalizations have numerous applications among others in the theory of differential equations and geometry (see for example [2–4,8,13,14,17,18]). Sometimes, in specific situations in applications, it is not known if it is satisfied for all arguments, so it is worth to investigate it on restricted domains to find out when a particular restricted domain has an effect on the form of the solutions and how.

Aczél in monograph [1], as a generalization of Eq. (1.2) considered the equation

$$f(x+y)+g(x-y)=2h(x)k(y)\quad x,y\in G.$$

In this paper we study the special case of the above equation, which we will call our second generalization of d'Alembert's functional equation i.e.,

$$f(x+y)+f(x-y)=2g(x)f(y)\quad x,y\in G \tag{1.3}$$

and we show analogous results for pairs of functions $f:G\to P,\ g:A\to P$ satisfying the equation

$$f(x+y)+f(x-y)=2g(x)f(y)\quad (x,y)\in A\times G, \tag{1.4}$$

where $(G,+)$ is an abelian group, A is a subgroup of G and $(P,+,\cdot)$ is a quadratically closed field with char $P\neq 2$ throughout this paper, unless explicitly stated otherwise.

In the present paper we obtain our results in a different way than in the paper [5].

For a function $m:G\to P$, m_e and m_o denote the even and odd parts of m, respectively, i.e., $m_e=\frac{1}{2}(m+m\circ\tau)$ and $m_o=\frac{1}{2}(m-m\circ\tau)$, where $\tau(x)=-x$ for $x\in G$. Moreover, for $a\in G$ and $D\subset G$ we write $2D:=\{2x:x\in D\}$, $a+D:=\{a+b:b\in D\}$, $a-D:=\{a-b:b\in D\}$ and $G/A:=\{[u]=u+A:u\in G\}$.

Let us yet recall a classical result which will be needed in our study (see [15, Theorem 2.2] and [10, Lemma 29.41] from which the last statement inferred).

Theorem 1.2. *Let $(B,+)$ be an abelian group and $\operatorname{char}P\neq 2$. A pair of functions $f,g:B\to P$ satisfies functional equation*

$$f(x+y)+f(x-y)=2f(x)g(y)\quad x,y\in B \tag{1.5}$$

if and only if $f(B)=\{0\}$ or there exists an exponential function $m:B\to P\setminus\{0\}$ such that $g=m_e$ and there are two possibilities:

(i) if $m=m\circ\tau$, i.e. if $m(B)\subseteq\{-1,1\}$, then f has the form

$$f(x)=m(x)(L(x)+\alpha)\quad x\in B, \tag{1.6}$$

where $L:B\to P$ is an additive function and $\alpha\in P$ is a constant,

(ii) if $m\neq m\circ\tau$, then f has the form

$$f(x)=\beta m_e(x)+\gamma m_o(x)\quad x\in B, \tag{1.7}$$

where $\beta,\ \gamma\in P$ are constants.

Moreover, m is uniquely determined by g, except that it may be interchanged with $m\circ\tau$.

2. Main result

The following theorem is the main result of this paper and gives the full description of all pairs of functions $f:G\to P, g:A\to P$ satisfying (1.4).

Theorem 2.1. *The pair of functions $f:G\to P$ and $g:A\to P$ is a solution of Eq. (1.4) if and only if $f(G)=\{0\}$ and g is arbitrary or*

(a) There exists an odd mapping $\gamma:G/2A\to P$ such that

$$f(y)=\gamma(y+2A),\ y\in G\quad\text{and}\quad g=0.$$

or

(b) There exist an exponential function $m:A\to\{-1,1\}$, a lifting $\xi:G/A\to G$, a family of additive functions $\mathcal{L}_\sigma:A\to P$ for $\sigma\in G/A$, and a function $\kappa:G/A\to P$ such that $g=m$ and

$$\xi([0])=0,\quad \xi([-y])=-\xi([y])\quad y\in G,\ [y]\neq[-y], \tag{2.1}$$