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Computation of matrix exponentials of special matrices ☆

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ABSTRACT

Computing matrix exponentials or transition matrices are important in signal processing and control systems. This paper studies the methods of computing the transition matrices related to linear dynamic systems, discusses and gives the properties of the transition matrices and explores the transition matrices of some special matrices. Some examples are provided to illustrate the proposed methods.

1. Introduction

In engineering, we often encounter algebraic equations or differential equations or decomposition of some special matrices [1–3]. In linear cases, these differential equations can be transformed into a set of the first-order differential equations of form $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$, where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is a real constant matrix, $\boldsymbol{x}(t) \in \mathbb{R}^n$ is a state vector. Solving these matrix differential equations leads to a matrix exponential $e^{\boldsymbol{A}t}$ of the transition matrix $\boldsymbol{A}$. Computing matrix functions plays an important role in science and engineering, including control theory. By applying mixed interpolation methods, Dehghan and Hajarian gave an algorithm for computing the matrix function $f(\boldsymbol{A})$ and proved its existence and uniqueness [4]. Wu discussed the matrix exponential of the matrix $\boldsymbol{A}$ which satisfies the relation $\boldsymbol{A}^{k+1} = \rho\boldsymbol{A}^k$ and gave explicit formulas for computing $e^{\boldsymbol{A}}$ [5].

Matrix functions or matrix exponentials have wide applications in signal filtering [6,7] and controller design [8,9]. Many different algorithms for computing the matrix exponentials have been reported. For example, Moore computed matrix exponentials by adopting the idea of expanding in either Chebyshev, Legendre or Laguerre orthogonal polynomials for the matrix exponentials [10]. Matrix operations are useful in linear algebra and matrix theory. Al Zhour and Kilicman discussed some different matrix products for partitioned and non-partitioned matrices and some useful connections of the matrix products [11], including Kronecker product, Hadamard product, block Kronecker product, block Hadamard product, block-matrix inner product (i.e., star product or ★ product) [12–14], etc. Dehghan and Hajarian presented an iterative method for solving the generalized coupled Sylvester matrix equations over generalized bisymmetric matrices and analyzed its performance [15,16].

In the area of system identification [17–21] and parameter estimation [22–24], e.g., the multi-innovation identification [25–27], one task is to estimate the parameter matrix/vector $\boldsymbol{A}$ and $\boldsymbol{b}$ of the state space model in (2) of the next section, and to determine the state vector $\boldsymbol{x}(t)$ which leads to the matrix exponential or transition matrix $e^{\boldsymbol{A}t}$. This paper studies the properties of the transition matrix and computes the transition matrices of some special matrices [3].

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The rest of the paper is organized as follows. Section 2 derives the solution of state equations and Section 3 discusses the state transition matrix and its properties. Section 4 computes the transition matrices of some special matrices. Section 5 provides an illustrative example to validate the proposed methods. Finally, Section 6 offers some concluding remarks.

2. Solution of state equations

This section introduces some basic facts to be used in the next sections, which can be found in Modern Control Theory textbook.

A continuous-time system described by a linear differential equation with constant coefficients can be transformed into a set of the first-order differential equations:

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_1u(t), \\ \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_2u(t), \\ \vdots \\ \dot{x}_3(t) = a_{n1}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_nu(t), \end{cases} \tag{1}$$

where t is a time variable, $u(t) \in \mathbb{R}$ is the input of the system, $x_i(t) \in \mathbb{R},\ i = 1, 2, \ldots, n$ are the state variables of the system.

Construct an n-dimensional state vector:

$$\boldsymbol{x}(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n.$$

Let

$$\boldsymbol{A} := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad \boldsymbol{b} := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n.$$

Eq. (1) can be written as a compact form,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{2}$$

When the input $u(t) = 0$, we obtain a homogeneous equation,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{3}$$

Here, we use the power series to solve this equation. Assume that the state vector has the form of

$$\boldsymbol{x}(t) = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 t + \boldsymbol{\alpha}_2 t^2 + \cdots + \boldsymbol{\alpha}_i t^i + \cdots, \tag{4}$$

where $\boldsymbol{\alpha}_i \in \mathbb{R}^n$ is the coefficient vector to be determined. Substituting $\boldsymbol{x}(t)$ into (3) gives

$$\begin{aligned} &\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 t + 3\boldsymbol{\alpha}_3 t^2 + \cdots + i\boldsymbol{\alpha}_i t^{i-1} + \cdots \\ &\quad = \boldsymbol{A}\boldsymbol{\alpha}_0 + \boldsymbol{A}\boldsymbol{\alpha}_1 t + \boldsymbol{A}\boldsymbol{\alpha}_2 t^2 + \cdots + \boldsymbol{A}\boldsymbol{\alpha}_i t^i + \cdots. \end{aligned}$$

Comparing the coefficients of the same power of both sides establishes equations:

$$\begin{cases} \boldsymbol{\alpha}_1 = \boldsymbol{A}\boldsymbol{\alpha}_0, \\ 2\boldsymbol{\alpha}_2 = \boldsymbol{A}\boldsymbol{\alpha}_1, \\ 3\boldsymbol{\alpha}_3 = \boldsymbol{A}\boldsymbol{\alpha}_2, \\ \vdots \\ i\boldsymbol{\alpha}_i = \boldsymbol{A}\boldsymbol{\alpha}_{i-1}. \end{cases} \Rightarrow \begin{cases} \boldsymbol{\alpha}_1 = \boldsymbol{A}\boldsymbol{\alpha}_0, \\ \boldsymbol{\alpha}_2 = \frac{1}{2}\boldsymbol{A}\boldsymbol{\alpha}_1 = \frac{1}{2!}\boldsymbol{A}^2\boldsymbol{\alpha}_0, \\ \boldsymbol{\alpha}_3 = \frac{1}{3}\boldsymbol{A}\boldsymbol{\alpha}_2 = \frac{1}{3!}\boldsymbol{A}^3\boldsymbol{\alpha}_0, \\ \vdots \\ \boldsymbol{\alpha}_i = \frac{1}{i}\boldsymbol{A}\boldsymbol{\alpha}_{i-1} = \frac{1}{i!}\boldsymbol{A}^i\boldsymbol{\alpha}_0. \end{cases}$$

Let $\boldsymbol{I}_n$ be an n-dimensional identity matrix and $\boldsymbol{I}$ be an identity matrix of appropriate sizes. Inserting $\boldsymbol{\alpha}_i$ into (4) gives

$$\boldsymbol{x}(t) = \left(\boldsymbol{I}_n + \boldsymbol{A}t + \frac{1}{2!}\boldsymbol{A}^2 t^2 + \frac{1}{3!}\boldsymbol{A}^3 t^3 + \cdots + \frac{1}{i!}\boldsymbol{A}^i t^i + \cdots\right)\boldsymbol{\alpha}_0.$$

Let $t = 0$, we have $\boldsymbol{\alpha}_0 = \boldsymbol{x}(0)$, the solution of the homogeneous equation in (3) is given by