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Certain new classes of Durrmeyer type operators

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ABSTRACT

In the present paper, we estimate the rate of convergence for functions having bounded derivatives for certain Durrmeyer type generalization of Jain and Pethe operators. In the last section, we also propose a new modification of the Lupaş operators and study some direct results. We also give some open problems for the readers.

1. Introduction

Motivated by the operators due to Jain and Pethe [12], in the year 2007 Abel and Ivan [1] considered the following operators

$$(S_{n,c}f)(x) = \sum_{v=0}^{\infty} p_{n,v}^{[c]}(x) f\left(\frac{v}{n}\right), x \geqslant 0 \tag{1.1}$$

where

$$p_{n,v}^{[c]}(x) = \left(\frac{c}{1+c}\right)^{ncx} \frac{(ncx)_v}{v!}(1+c)^{-v} (v = 0, 1, \ldots)$$

where the Pochhammer symbol $(n)_k$ is defined as

$$(n)_k = n(n+1)(n+2)\ldots.(n+k-1),$$

with $(n)_0 = 1$. As the special values i.e. if $c \to \infty$, the operators (1.1) reduce to the classical Szász-Mirakyan operators and if $c = 1$, we get the Lupaş operators [13]. Abel and Ivan [1] estimated the complete asymptotic expansion of the operators $(S_{n,c}f)$. Agratini [2] estimated some approximation properties of the operators $(S_{n,c}f)(x)$, and Tarabie [15] defined Beta type modifications of these operators and studied statistical convergence. In order to approximate Lebesgue integrable functions on the interval $[0, \infty)$, very recently Gupta [9] proposed a Durrmeyer type modification of the operators (1.1), by considering the weight functions $b_{n,v}^{[d]}(t)$ as follows:

$$(D_{n,c,d}f)(x) \equiv D_{n,c,d}(f, x) = (n-d) \sum_{v=0}^{\infty} p_{n,v}^{[c]}(x) \int_0^{\infty} b_{n,v}^{[d]}(t) f(t) dt, x \geqslant 0 \tag{1.2}$$

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where $b^{[d]}_{n,\upsilon}(t) = (-1)^{\upsilon}\frac{t^{\upsilon}}{\upsilon!}\phi^{(\upsilon)}_{n,d}(t)$. Here $\{\phi_{n,d}(x)\}_{n=1}^{\infty}$ is a sequence of functions, defined on $[0,b](b>0)$, and satisfying the following properties for every $n\in\mathbf{N}$ and $\upsilon\in\mathbf{N}\cup\{0\}$:

(i) $\phi_{n,d}\in C^{\infty}([a,b])(b>a\geqslant 0)$;
(ii) $\phi_{n,d}(0)=1$;
(iii) $\phi_{n,d}(x)$ is completely monotone, that is $(-1)^{\upsilon}\phi^{(\upsilon)}_{n,d}(x)\geqslant 0\ (0\leqslant x\leqslant b)$;
(iv) there exists an integer d such that

$$\phi^{(\upsilon+1)}_{n,d}(x) = -n\phi^{(\upsilon)}_{n+d,d}(x),\quad (n>\max\{0,-d\};\ x\in[0,b]).$$

By simple computation it can easily be verified that $\sum_{\upsilon=0}^{\infty}p^{[c]}_{n,\upsilon}(x)=1$ and $\int_0^{\infty}b^{[d]}_{n,\upsilon}(t)dt = 1/(n-d)$. Such type of values of $b^{[d]}_{n,\upsilon}(t)$ were considered in [14], while introducing some other summation-integral type generalized operators. Some of the well known operators can also be expressed in terms of Hypergeometric functions, which gives the new direction to think the applications of approximation theory in special functions. As is well known, the hypergeometric function is given by

$$_2F_1(a,b;c;x) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k$$

and the confluent Hypergeometric function by

$$_1F_1(a;b;x) = \sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k k!}x^k.$$

We represent here the special cases of the operators (1.2) in terms of Hypergeometric functions as follows:

(1) If $d=0, \phi_{n,0}(t)=e^{-nt}$, then $b^{[0]}_{n,\upsilon}(t)=e^{-nt}\frac{(nt)^{\upsilon}}{\upsilon!}$ and the operators can be represented as

$$\begin{aligned}(D_{n,c,0}f)(x) &= n\sum_{\upsilon=0}^{\infty}p^{[c]}_{n,\upsilon}(x)\int_0^{\infty}e^{-nt}\frac{(nt)^{\upsilon}}{\upsilon!}f(t)dt\\ &= n\left(\frac{c}{1+c}\right)^{ncx}\sum_{\upsilon=0}^{\infty}\frac{(ncx)_{\upsilon}}{\upsilon!(1+c)^{\upsilon}}\int_0^{\infty}e^{-nt}\frac{(nt)^{\upsilon}}{\upsilon!}f(t)dt\\ &= n\left(\frac{c}{1+c}\right)^{ncx}\int_0^{\infty}e^{-nt}f(t)\sum_{\upsilon=0}^{\infty}\frac{(ncx)_{\upsilon}}{(1)_{\upsilon}\upsilon!}\left(\frac{nt}{1+c}\right)^{\upsilon}dt\\ &= n\left(\frac{c}{1+c}\right)^{ncx}\int_0^{\infty}e^{-nt}f(t)\,{}_1F_1\left(ncx;1;\frac{nt}{1+c}\right).\end{aligned}$$

(2) If $d=1,\ \phi_{n,1}(t)=(1+t)^{-n}$, then clearly $b^{[1]}_{n,\upsilon}(t)=\binom{n+\upsilon-1}{\upsilon}\frac{t^{\upsilon}}{(1+t)^{n+\upsilon}}$ and the operators can be represented as

$$\begin{aligned}(D_{n,c,1}f)(x) &= (n-1)\int_0^{\infty}\left(\frac{c}{1+c}\right)^{ncx}\sum_{\upsilon=0}^{\infty}\frac{(ncx)_{\upsilon}}{\upsilon!(1+c)^{\upsilon}}\frac{(n)^{\upsilon}}{\upsilon!}\frac{t^{\upsilon}}{(1+t)^{n+\upsilon}}f(t)dt\\ &= (n-1)\left(\frac{c}{1+c}\right)^{ncx}\int_0^{\infty}(1+t)^{-n}f(t)\sum_{\upsilon=0}^{\infty}\frac{(ncx)_{\upsilon}(n)_{\upsilon}}{(1)_{\upsilon}\upsilon!}\left(\frac{t}{(1+c)(1+t)}\right)^{\upsilon}dt\\ &= (n-1)\left(\frac{c}{1+c}\right)^{ncx}\int_0^{\infty}(1+t)^{-n}f(t)\,{}_2F_1\left(ncx,n;1;\frac{t}{(1+c)(1+t)}\right).\end{aligned}$$

Gupta [9] estimated the direct results in terms of modulus of continuity and an asymptotic formula in ordinary approximation for the operators (1.2). We now extend the studies on the operators (1.2). In the second section, we present some basic lemmas concerning moments of the operators (1.2). In section three, we establish the rate of convergence for functions having bounded derivatives. In the last section we propose another generalization of similar operators and estimate some direct results.

2. Basic results

We need the following results in the sequel:

Lemma 1 [9]. *Let* $e_r(t)=t^r,\ r\geqslant 0$, *then we have*

$$(D_{n,c,d}e_r)(x) = \frac{\Gamma(r+1)}{\prod_{i=1}^{r+1}(n-id)}(n-d)\left(\frac{c}{1+c}\right)^{ncx}{}_2F_1\left(ncx,r+1;1;\frac{1}{1+c}\right).$$

Further, we have

$$(D_{n,c,d}e_0)(x)=1,(D_{n,c,d}e_1)(x)=\frac{1+nx}{n-2d}$$