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Differential subordinations involving arithmetic and geometric means

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ABSTRACT

The paper concerns the differential subordination with the expression combined by arithmetic and geometric means:

$$\alpha[p(z)]^{\delta}+(1-\alpha)\left[p(z)+\frac{zp'(z)}{p(z)}\right]^{\mu}\prec\varphi(z),\quad (p(0)=\varphi(0)=1,\ |z|<1),$$

where δ,μ and α are real numbers such that $\delta,\mu\in\langle 1,2\rangle$, $\alpha\in\langle 0,1\rangle$. For $\delta\in\langle 1,2\rangle$, $\mu\in\langle 0,1\rangle$, $\alpha\in\langle 0,1\rangle$ we also study the differential subordination

$$\alpha[p(z)]^{\delta}+(1-\alpha)[p(z)]^{\mu}\left[p(z)+\frac{zp'(z)}{p(z)}\right]^{1-\mu}\prec\varphi(z),\quad (p(0)=\varphi(0)=1,\ |z|<1).$$

Several applications of the studied subordination in the theory of analytic functions are given.

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1. Introduction

Let $\mathcal{U}=\{z\in\mathbb{C}:|z|<1\}$ denote the open unit disk of the complex plane, and $\mathcal{A}$ be the class of all analytic functions in $\mathcal{U}$, with the Taylor series expansion:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n.$$

If f and g are two functions analytic in $\mathcal{U}$, then f is said to be subordinate to g, written as $f\prec g$ or $f(z)\prec g(z)$, if there exists a Schwarz function ω (i.e. analytic in $\mathcal{U}$, with $\omega(0)=0$ and $|\omega(z)|<1, z\in\mathcal{U}$) such that $f(z)=g(\omega(z))$. In particular, if g is univalent in $\mathcal{U}$, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(\mathcal{U})\subset g(\mathcal{U})$.

The arithmetic means of some functions and expressions is very frequently used in mathematics, specially in geometric functions theory. Making use of the arithmetic means Mocanu [11] introduced the class of α-convex ($0\leqslant\alpha\leqslant 1$) functions as follows

$$\mathcal{M}_{\alpha}=\left\{f\in\mathcal{A}:\Re\left[(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>0,\ z\in\mathcal{U}\right\},$$

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which, in some case, proclaims the class of starlike, and in the other, convex functions. In general, the class of α-convex functions determines the *arithmetic bridge* between starlikeness and convexity. Subsequently, several papers involving arithmetic means were published, see for example [1,13,14].

Using the geometric means, Lewandowski et al. [9] defined the class of μ-starlike functions ($\mu \in \langle 0, 1 \rangle$) consisting of the functions $f \in \mathcal{A}$ that satisfy the inequality

$$\Re\left[\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\mu}\right] > 0 \quad (z \in \mathcal{U}).$$

We note that the class of μ-starlike functions constitutes the *geometric bridge* between starlikeness and convexity. Setting $p(z) := zf'(z)/f(z)$, the above inequality is equivalent to

$$\Re\left[p(z)^{\mu}\left(p(z)+\frac{zp'(z)}{p(z)}\right)^{1-\mu}\right] > 0,$$

which was shown to imply $\Re\, p(z) > 0$, so μ-starlike functions are starlike.

Following these ideas, Kanas et al. [2] investigated the subordination

$$p(z)\left(1+\frac{zp'(z)}{p(z)}\right)^{\alpha} \prec h(z) \quad (p(0) = h(0) = 0),$$

while Kim and Lecko [7] found the largest constant $C(\alpha)$ such that the subordination

$$p(z)\left(1+\frac{zp'(z)}{p(z)}\right)^{\alpha} \prec C(\alpha)h(z) \quad (p(0) = h(0)),$$

implies $p(z) \prec h(z)$. For some other interesting results on differential subordination involving arithmetic or geometric means, see also [4,3,5,6,8].

The purpose of this paper is interconnecting and unifying both attempt: arithmetic and geometric; containing known and indicating new fields of study. Among other things this approach brings new elements of geometry connections between functionals in geometric function theory. In order to prove our main results, we need the following lemmas:

Lemma 1.1 [10, p. 24]. *Let $q(z) = 1 + q_1 z^n + \cdots$ be analytic and univalent on $\mathcal{U}$, injective on $\partial\mathcal{U} \setminus \{1\}$ and such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} \setminus \{1\}$. Let p be an analytic functions in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If p is not subordinate to q, then there exist the points $z_0 \in \mathcal{U}$ and $\zeta_0 \in \partial\mathcal{U} \setminus \{1\}$, and $m \geqslant n \geqslant 1$ for which*

$$p(\mathcal{U}_{|z_0|}) \subset q(\mathcal{U}_{|z_0|}), \quad p(z_0) = q(\zeta_0) \quad \text{and} \quad z_0 p'(z_0) = m\zeta_0 q'(\zeta_0).$$

Lemma 1.2 [10, p. 26]. *Let p be an analytic function in $\mathcal{U}$ such that $p(0) = 1$, $p(z) \not\equiv 1$. If $z_0 \in \mathcal{U}$ satisfies*

$$\Re\, p(z_0) = \min\{\Re\, p(z) : |z| \leqslant |z_0|\},$$

then

$$z_0 p'(z_0) \leqslant -\frac{|p(z_0) - 1|^2}{2\Re[1 - p(z_0)]}.$$

Below, we give a lemma due to Nunokawa [12], only in a slightly different but equivalent form, which is more convenient for our next consideration.

Lemma 1.3 [12]. *Let p be an analytic function in $\mathcal{U}$ such that $p(0) = 1$, $p(z) \not\equiv 1$. If $z_0 \in \mathcal{U}$ satisfies*

$$|\arg p(z_0)| = \max\{\arg p(z) : |z| \leqslant |z_0|\} = \gamma\frac{\pi}{2}, \quad \text{and} \quad p(z_0) = (ix)^{\gamma},$$

then

$$|\arg[z_0 p'(z_0)]| = (\gamma + 1)\frac{\pi}{2} \quad \text{and} \quad |z_0 p'(z_0)| = \left|\frac{\gamma x^{\gamma}}{2}\left(x + \frac{1}{x}\right)\right|.$$

2. Arithmetic–geometric subordination results

In this section we study some differential subordinations that involve the arithmetic and geometric means. We use the methods of differential subordination theory, introduced and developed by Miller and Mocanu (see [10] for comprehensive study). In the sequel we assume that all the powers takes its principal branches.

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