Applied Mathematics and Computation 222 (2013) 231–243

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Estimation of bounds for the zeros of a polynomial using numerical radius ☆

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ARTICLE INFO

Keywords:
Numerical radius
Companion matrix
Zeros of polynomial

ABSTRACT

The estimation of zeros of a polynomial have been done by many mathematicians over the years using various approaches. In this paper we estimate the upper bound for the zeros of a given polynomial using Hilbert space technique involving Frobenius companion matrix and numerical radius. We first obtain numerical range and numerical radius for certain class of matrices and use them to estimate the bounds for zeros of a given polynomial. We illustrate with examples to show that the estimations obtained here is better than the previously known estimations. We also obtain a sequence of real numbers which converges exactly to the spectral radius of some special class of matrices.

1. Introduction

Consider a monic polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots a_1 z + a_0$ where $a_0, a_1, \ldots a_{n-1}$ are all scalars, may be complex or real. For $n = 1, 2, 3, 4$, we can exactly compute the zeros of the polynomial in terms of coefficients a_i and their radicals, the degree of difficulty of the computation of zeros increases with the degree of the polynomials from 1 to 4. For $n = 5$ or more there is no general method (in fact, by Galois theory there cannot be any such method for computation of zeros involving the coefficients and their radicals) by which one can compute the zeros of the polynomial using the coefficients and their radicals, so an estimation of the bounds for the zeros of a polynomial becomes more important and useful. The need for estimation of zeros of a polynomial arises frequently in various areas of applications. In fact one of the most fundamental problem of numerical mathematics is the estimation for zeros of a polynomial.

The zeros of a given polynomial have been estimated by many mathematicians over the years using different approaches. To mention a few of them are Fujiwara [6], Carmichael and Mason [7], Montel [12,13], Cauchy [7], Fujii and Kubo [4,5], Linden [9,10], Alpin et al. [1], Kittaneh [8], Paul and Bag [14], Dehmer and Mowshowitz [3], Mignotte and Stefanescu [11], Stefanescu [16].

Recently in [14] we used Hilbert space technique involving numerical radius and companion matrix to find better estimation for upper bounds for zeros of a given polynomial. The book [2] by Debnath and Mikusinski is a nice reference for introduction to Hilbert space theory and for information related to companion matrix and related things one can look at the book [7] by Horn and Johnson.

Suppose $A \in M_n(\mathbb{C})$. Let W(A), $\sigma(A)$ denote respectively the numerical range, spectrum of A and $w(A), r_\sigma(A)$ denote respectively the numerical radius, spectral radius of A, i.e.,

☆ The research of second author is supported by CSIR, India.
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http://dx.doi.org/10.1016/j.amc.2013.07.046

$W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}$
$w(A) = \sup\{|\lambda| : \lambda \in W(A)\}$
$\sigma(A) = \{\lambda : \lambda \text{ is an eigenvalue of } A\}$ and
$r_\sigma(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$.

It is well-known that

$$\frac{\|A\|}{2} \leqslant w(A) \leqslant \|A\|.$$

Kittaneh [8] improved on the second inequality to prove that

$$w(A) \leqslant \frac{1}{2}\|A\| + \frac{1}{2}\|A^2\|^{\frac{1}{2}}$$

and used it to estimate zeros of a given polynomial.

Paul and Bag [15] recently conjectured that

$$w(A) \leqslant \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \|A^{2^n}\|^{\frac{1}{2^n}}.$$

In this paper we first obtain the numerical radius of a matrix $A \in M_n(\mathbb{C})$ satisfying $A^2 - (\lambda_1 + \lambda_2)A + \lambda_1\lambda_2 I = 0$, where λ_1, λ_2 are scalars, real or complex. Then we obtain inequalities involving numerical radius of $A \in M_n(\mathbb{C})$ and use them to estimate the bounds for zeros of a given polynomial. We also give an iterative method to generate a sequence of positive real numbers that will converge to the spectral radius of a given matrix. We finally show with examples that the estimation obtained by our method is better than previously known estimations.

2. On numerical radius of a matrix

We obtain the numerical radius of a matrix $A \in M_n(\mathbb{C})$ satisfying $A^2 - (\lambda_1 + \lambda_2)A + \lambda_1\lambda_2 I = 0$, where λ_1, λ_2 are scalars, real or complex.

Theorem 2.1. *Suppose* $A \in M_n(\mathbb{C})$ *with* $A^2 - (\lambda_1 + \lambda_2)A + \lambda_1\lambda_2 I = 0$, *where* λ_1, λ_2 *are scalars, real or complex. Then the numerical range of A is an ellipse with foci at* λ_1 *and* λ_2, *major axis* $a = \frac{1}{2}\left(\sqrt{|\lambda_1 - \lambda_2|^2 + |b|^2}\right)$, *minor axis* $\frac{1}{2}|b|$ *and the major axis has an inclination* $\theta = \arg\frac{1}{2}(\lambda_1 - \lambda_2)$ *with the real axis.*

Proof. To prove this theorem we need the following lemma:

Lemma 2.2. *Suppose* $T \in M_n(\mathbb{C})$ *and*

$$T = \begin{pmatrix} I_r & B_{r,n-r} \\ O_{n-r,r} & -I_{n-r} \end{pmatrix}.$$

The numerical range of the operator T is an ellipse with centre at $(0, 0)$, major axis $\sqrt{1 + \frac{1}{4}\|B\|^2}$ and minor axis $\frac{1}{2}\|B\|$.

Proof of lemma. Let $a = \sqrt{1 + \frac{1}{4}\|B\|^2}, b = \frac{1}{2}\|B\|$ and

$$E = \{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1\}.$$

We first prove that $W(T) \subset E$.

Let $\lambda = \langle TZ, Z\rangle \in W(T)$ where $\|Z\| = 1$.

$$\textit{Take } Z = \begin{pmatrix} X \\ Y \end{pmatrix}, X = U\cos\theta \quad \text{and} \quad Y = V\sin\theta.$$

Then $\lambda = \cos 2\theta + \frac{1}{2}\langle BV, U\rangle \sin 2\theta$. If $\lambda = x + iy$ and $\phi = \arg\langle BV, U\rangle$ then we get

$$x = \cos 2\theta + \frac{1}{2}|\langle BV, U\rangle| \sin 2\theta \cos\phi \quad \text{and} \quad y = \frac{1}{2}|\langle BV, U\rangle| \sin 2\theta \sin\phi.$$

Therefore

$$(x - \cos 2\theta)^2 + y^2 = \frac{1}{4}|\langle BV, U\rangle|^2 \sin^2 2\theta,$$