Applied Mathematics and Computation 222 (2013) 478–489

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the elements $aa^\dagger$ and $a^\dagger a$ in a ring

Julio Benítez [a,*], Dragana Cvetković-Ilić [b,1]

[a] Instituto de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain
[b] Faculty of Sciences and Mathematics, Department of Mathematics, University of Nis, 18000 Nis, Serbia

ARTICLE INFO

Keywords:
Ring with involution
Projections
Moore–Penrose inverses
Principal ideals
Annihilators

ABSTRACT

We study various functions, principal ideals and annihilators depending on the projections $aa^\dagger$ and $a^\dagger a$ for a Moore–Penrose invertible ring element, extending recent work of O.M. Baksalary and G. Trenkler for matrices.

1. Introduction

Throughout this paper, the symbol $\mathcal{R}$ will denote a unital ring (1 will be its unit) with an involution. Let us recall that an *involution* in a ring $\mathcal{R}$ is a map $a \mapsto a^*$ in $\mathcal{R}$ such that $(a+b)^* = a^* + b^*$, $(ab)^* = b^* a^*$ and $(a^*)^* = a$, for any $a, b \in \mathcal{R}$.

We say that $a \in \mathcal{R}$ is *regular* if there exists $b \in \mathcal{R}$ such that $aba = a$. It can be proved that for any $a \in \mathcal{R}$, there is at most one $a^\dagger \in \mathcal{R}$ (called the *Moore–Penrose inverse* of a) such that

$$aa^\dagger a = a, \quad a^\dagger a a^\dagger = a^\dagger, \quad (aa^\dagger)^* = aa^\dagger, \quad (a^\dagger a)^* = a^\dagger a.$$

In [9] it was proved that any complex matrix has a unique Moore–Penrose inverse, however, let us notice that the proof given therein is valid to guarantee the uniqueness – if the Moore–Penrose inverse exists – in a ring with involution. If there exists such $a^\dagger$ we will say that a is *Moore–Penrose invertible*. The subset of $\mathcal{R}$ composed of all Moore–Penrose invertible elements will be denote by $\mathcal{R}^\dagger$. We write $\mathcal{R}^{-1}$ for the set of all invertible elements in $\mathcal{R}$. The word *projection* will be reserved for an element q of $\mathcal{R}$ which is self-adjoint and idempotent, that is $q^* = q = q^2$. A ring $\mathcal{R}$ is called $*$-*reducing* if every element a of $\mathcal{R}$ obeys the implication $a^*a = 0 \Rightarrow a = 0$.

Let $x \in \mathcal{R}$ and let $p \in \mathcal{R}$ be an idempotent ($p = p^2$). Then we can write

$$x = pxp + px(1-p) + (1-p)xp + (1-p)x(1-p)$$

and use the notations

$$x_{11} = pxp, \quad x_{12} = px(1-p), \quad x_{21} = (1-p)xp, \quad x_{22} = (1-p)x(1-p).$$

Every projection $p \in \mathcal{R}$ induces a matrix representation which preserves the involution in $\mathcal{R}$, namely $x \in \mathcal{R}$ can be represented by means of the following matrix:

$$x = \begin{bmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-p) \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}. \tag{1.1}$$

* Corresponding author.
E-mail addresses: jbenitez@mat.upv.es (J. Benítez), dragana@pmf.ni.ac.rs (D. Cvetković-Ilić).
1 Supported by Grant No. 174007 of the Ministry of Science, Technology and Development, Republic of Serbia.

0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.07.015

The purpose of this paper is to study several ideals involving the projections $aa^\dagger$ and $a^\dagger a$, when $a \in \mathcal{R}^\dagger$. We shall consider two kinds of ideals. The *principal ideals* (also called image ideals) generated by $b \in \mathcal{R}$ are defined by $b\mathcal{R} = \{bx : x \in \mathcal{R}\}$ and $\mathcal{R}b = \{xb : x \in \mathcal{R}\}$. The *annihilators* (also called kernel ideals) of $b \in \mathcal{R}$ are defined by $b^\circ = \{x \in \mathcal{R} : bx = 0\}$ and ${}^\circ b = \{x \in \mathcal{R} : xb = 0\}$. If $\mathcal{R}$ is a ring with the unit and $p \in \mathcal{R}$, then it is quickly seen that $p\mathcal{R}p = \{pxp : x \in \mathcal{R}\}$ is a sub-ring whose unity is p. From now on, for an arbitrary projection p, we shall denote $\overline{p} = 1 - p$.

The following elementary lemma will be many times used in the sequel.

Lemma 1.1. *Let $\mathcal{R}$ be a ring with involution and $a \in \mathcal{R}$. Then*

(i) $a \in \mathcal{R}^\dagger \iff a^* \in \mathcal{R}^\dagger$. *Furthermore,* $(a^*)^\dagger = (a^\dagger)^*$.
(ii) *If* $a \in \mathcal{R}^\dagger$, *then* $a^\dagger \in \mathcal{R}^\dagger$ *and* $(a^\dagger)^\dagger = a$.
(iii) *If* $a \in \mathcal{R}^\dagger$, *then* $a^*a, aa^* \in \mathcal{R}^\dagger$ *and*

$$(a^*a)^\dagger = a^\dagger(a^*)^\dagger, \quad (aa^*)^\dagger = (a^*)^\dagger a^\dagger, \quad a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger, \quad a^* = a^\dagger aa^* = a^*aa^\dagger.$$

(iv) *If $\mathcal{R}$ is $*$-reducing, then* $a^*a \in \mathcal{R}^\dagger \Rightarrow a \in \mathcal{R}^\dagger$ *and* $aa^* \in \mathcal{R}^\dagger \Rightarrow a \in \mathcal{R}^\dagger$.

Proof. It is evident that (i)–(iii) hold. We will prove only the first implication of (iv) since to prove the other one, it is sufficient to make the same argument for a^* instead of a. Assume that $a^*a \in \mathcal{R}^\dagger$ and let $x = (a^*a)^\dagger a^*$. Observe that the Moore–Penrose inverse of a selfadjoint Moore–Penrose invertible element is again self-adjoint, and thus, $(a^*a)^\dagger$ is self-adjoint. Now $(ax)^* = \left[a(a^*a)^\dagger a^*\right]^* = a(a^*a)^\dagger a^* = ax; xa = (a^*a)^\dagger a^*a$ is selfadjoint; $xax = (a^*a)^\dagger a^*a(a^*a)^\dagger a^* = (a^*a)^\dagger a^* = x$. Finally, $a^*axa = a^*a(a^*a)^\dagger a^*a = a^*a$, and since $\mathcal{R}$ is $*$-reducing, we get $axa = a$. □

A consequence of Lemma 1.1 is that

$$\text{if } x \in \mathcal{R}^\dagger \text{ is self-adjoint, then } xx^\dagger = x^\dagger x. \tag{1.2}$$

For the class of Moore–Penrose invertible elements $x \in \mathcal{R}$ such that $xx^\dagger = x^\dagger x$, the reader is referred to [3].

2. Group inverses

Let $\mathcal{R}$ be a ring (possibly without an involution). If $a \in \mathcal{R}$, then there is at most one $x \in \mathcal{R}$ such that

$$axa = a, \quad xax = x, \quad ax = xa.$$

When such x exists, we will write $x = a^\#$ and we will say that x is the *group inverse* of a and a is *group invertible*. The symbol $\mathcal{R}^\#$ will denote the set of all group invertible elements of $\mathcal{R}$.

In this paragraph, let F be a square complex matrix. In [1, p. 10215] it was given a list of several equivalent conditions (involving the orthogonal projectors $FF^\dagger$ and $F^\dagger F$) for F to has the group inverse. The proof given therein relies in rank matrix theory and a matrix decomposition given by Hartwig and Spindelböck [4]. However, as we shall see, many of these equivalences can be stated in a ring setting, and proved by algebraic reasonings.

We shall use the following result [10, Prop. 8.22], whose proof is included for the convenience of the reader and which implies that in a commutative ring, group invertibility is the same as the existence of a generalized inverse.

Theorem 2.1. *Let $\mathcal{R}$ be a unital ring and $a \in \mathcal{R}$. Then a is group invertible if and only if there exist $x, y \in \mathcal{R}$ such that $a^2x = a$ and $ya^2 = a$.*

Proof. If $a \in \mathcal{R}^\#$ we have $a^2a^\# = a = a^\#a^2$.

Reciprocally, assume that there exist $x, y \in \mathcal{R}$ such that $a^2x = a$ and $ya^2 = a$. We will prove $yax = a^\#$. First, let us see that $ax = ya^2x = ya$. Now, $a(yax) = a(ya)x = a^2x^2 = ax$ and $(yax)a = y(ax)a = y^2a^2 = ya$ implies that $a(yax) = (yax)a$. Finally $a(yax)a = ya^2 = a$ and $(yax)a(yax) = yayax = yax$. □

Obviously, Theorem 2.1 implies that in a commutative ring, group invertibility is the same as regularity.

Observe that under the hypothesis of Theorem 2.1, one has

$$a^2x = a \text{ and } ya^2 = a \ \Rightarrow \ a^\# = yax. \tag{2.1}$$

Let us notice that by Theorem 2.1 one can deduce that for $a \in \mathcal{R}$,

$$a \in \mathcal{R}^\# \iff a\mathcal{R} = a^2\mathcal{R} \text{ and } \mathcal{R}a = \mathcal{R}a^2.$$

This latter equivalence can be viewed as a ring version of "for a matrix $F \in \mathbb{C}_{n,n}$, there exists $F^\#$ if and only if $\text{rank}(F^2) = \text{rank}(F)$" (see [5, Section 4.4]).

It was mentioned in [1, p. 10215] that for a given square complex matrix F, there exists $F^\#$ if and only if $\mathcal{R}(F) \cap \mathcal{N}(F) = \{0\}$, where $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denotes, respectively, the column space and the null space of a matrix. Let us notice