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# On efficient weighted-Newton methods for solving systems of nonlinear equations

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ABSTRACT

In this study, we present iterative methods of convergence order four and six for solving systems of nonlinear equations. The fourth order scheme is composed of two steps, namely; Newton iteration as the first step and weighted-Newton iteration as the second step. The sixth order scheme is composed of three steps; the first two steps are same as that of fourth order scheme whereas the third step is again based on weighted-Newton iteration. Computational efficiency in its general form is discussed. Comparison between the efficiencies of proposed techniques and existing techniques is made. It is proved that for large systems the new methods are more efficient. Numerical tests are performed, which confirm the theoretical results. Moreover, theoretical order of convergence is verified in the examples.



## 1. Introduction

Solving the system $F(x) = 0$ of nonlinear equations is a common and important problem in science and engineering [1–3], that is, for a given nonlinear function $F(x) : D \subset R^m \longrightarrow R^m$, where $F(x) = (f_1(x),\ f_2(x), \ldots, f_m(x))^t$ and $x = (x_1, x_2, \ldots, x_m)^t$, to find a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^t$ such that, $F(\alpha) = 0$. The most widely used method for this purpose is the classical Newton's method [3,4] which converges quadratically under the conditions that the function $F$ is continuously differentiable and a good initial approximation $x^{(0)}$ is given. It is defined by

$$x^{(k+1)} = \mathrm{M}_2(x^{(k)}) = x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}), \quad (k = 0, 1, \ldots), \tag{1}$$

where $F'(x^{(k)})^{-1}$ is the inverse of Fréchet derivative $F'(x^{(k)})$ of the function $F(x)$.

In order to improve the order of convergence of Newton's method, several methods have been proposed in literature, see, for example [5–11] and references therein. In particular, the methods proposed by Grau-Sánchez et al. [11] with fourth and sixth order convergence are noteworthy. The fourth order methods are defined as

$$\begin{aligned} y^{(k)} &= \mathrm{M}_2(x^{(k)}), \\ x^{(k+1)} &= \mathrm{M}_{4,1}(x^{(k)}, y^{(k)}) = y^{(k)} - (2\,[y^{(k)}, x^{(k)}; F] - F'(x^{(k)}))^{-1}F(y^{(k)}) \end{aligned} \tag{2}$$

and

$$\begin{aligned} y^{(k)} &= \mathrm{M}_2(x^{(k)}), \\ x^{(k+1)} &= \mathrm{M}_{4,2}(x^{(k)}, y^{(k)}) = y^{(k)} - (2\,[y^{(k)}, x^{(k)}; F]^{-1} - F'(x^{(k)})^{-1})F(y^{(k)}), \end{aligned} \tag{3}$$

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where $[y^{(k)},x^{(k)};F]$ is first order divided difference of $F$ and $[y^{(k)},x^{(k)};F]^{-1}$ is inverse of this divided difference. The scheme (2) is the generalization of Ostrowski's method [1] whereas (3) is the generalization of the method introduced in [12]. It is clear that both schemes utilize two functions, one derivative, one divided difference and two matrix inversions per iteration.

The sixth order methods, which are based on the above methods, are given by (see [11])

$$\begin{aligned} y^{(k)} &= \mathrm{M}_2(x^{(k)}),\\ z^{(k)} &= \mathrm{M}_{4,1}(x^{(k)},y^{(k)}),\\ x^{(k+1)} &= \mathrm{M}_{6,1}(x^{(k)},y^{(k)},z^{(k)}) = z^{(k)} - (2[y^{(k)},x^{(k)};F] - F'(x^{(k)}))^{-1}F(z^{(k)}) \end{aligned} \tag{4}$$

and

$$\begin{aligned} y^{(k)} &= \mathrm{M}_2(x^{(k)}),\\ z^{(k)} &= \mathrm{M}_{4,2}(x^{(k)},y^{(k)}),\\ x^{(k+1)} &= \mathrm{M}_{6,2}(x^{(k)},y^{(k)},z^{(k)}) = z^{(k)} - (2[y^{(k)},x^{(k)};F]^{-1} - F'(x^{(k)})^{-1})F(z^{(k)}). \end{aligned} \tag{5}$$

Per iteration both methods use three functions, one derivative, one divided difference and two matrix inversions. These algorithms are notable not only for their simplicity but also for efficient character.

In this study, we shall follow the basic principle of numerical analysis that a genuine ranking of numerical algorithms can be attained using computational efficiency that is always proportional to quality of an algorithm and inversely proportional to its computational cost. Quality of an algorithm concerns with the convergence speed of algorithm along with its structure. Computational cost concerns with the amount of calculation work required to evaluate functions, derivatives, matrix inversions during the entire process. Taking into account these considerations, we here devise a fourth order scheme which requires the same number of evaluations as the fourth order schemes (2) and (3) with the exception that instead of two matrix inversions here we use only one. Furthermore, based on this scheme a sixth order scheme is proposed with an additional function evaluation which also uses one matrix inversion.

The rest of the paper is organized as follows. Section 2 includes development of the fourth order method and its analysis of convergence. In Section 3, sixth order method with its analysis of convergence is presented. The computational efficiency in its general form is discussed and compared with the existing methods in Section 4. In Section 5, various numerical examples are considered to confirm the theoretical results and a comparison with the existing similar methods is also presented. Concluding remarks are given in Section 6.

## 2. The fourth order method

As mentioned above an efficient iterative method is one which has higher convergence order with minimum computational cost. While solving systems of nonlinear equations by an iterative method a long and tedious calculation work is required for the evaluation of inverse of a matrix, which becomes a barrier in the development of an efficient iterative scheme. Thus, it will turn out to be judicious if we use as small number of such inversions as possible. Keeping this in mind we consider the following scheme:

$$\begin{aligned} y^{(k)} &= \mathrm{M}_2(x^{(k)}),\\ x^{(k+1)} &= y^{(k)} - (aI + bF'(x^{(k)})^{-1}[y^{(k)},x^{(k)};F])F'(x^{(k)})^{-1}F(y^{(k)}), \end{aligned} \tag{6}$$

where $\mathrm{M}_2(x^{(k)})$ is the Newton iteration (1); $a$ and $b$ are parameters to be determined; and $I$ is identity matrix. We call this scheme the weighted-Newton method because of the second step being weighted-Newton step.

In order to discuss behavior of the scheme (6) we consider the following formula of divided difference operator $[\cdot,\cdot;F] : D\times D \subset R^m \times R^m \longrightarrow L(R^m)$ (see [2,11,13])

$$[x+h,x;F] = \int_0^1 F'(x+th)\,dt,\ \forall x,h \in R^m. \tag{7}$$

Expanding $F'(x+th)$ in Taylor series at the point $x$ and integrating, we have

$$[x+h,x;F] = \int_0^1 F'(x+th)\,dt = F'(x) + \frac{1}{2}F''(x)h + \frac{1}{6}F'''(x)h^2 + O(h^3). \tag{8}$$

where $h^i = (h,h,\overset{i}{\cdots},h)$, $h \in R^m$.

Let $e^{(k)} = x^{(k)} - \alpha$. Developing $F(x^{(k)})$ in a neighborhood of $\alpha$ and assuming that $\Gamma = F'(\alpha)^{-1}$ exists, we have

$$F(x^{(k)}) = F'(\alpha)(e^{(k)} + A_2(e^{(k)})^2 + A_3(e^{(k)})^3 + A_4(e^{(k)})^4 + O((e^{(k)})^5)), \tag{9}$$

where $A_i = \frac{1}{i!}\Gamma F^{(i)}(\alpha) \in L_i(R^m,R^m)$ and $(e^{(k)})^i = (e^{(k)},e^{(k)},\overset{i}{\cdots},e^{(k)})$, $e^{(k)} \in R^m$.

Also,