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Varying discrete Laguerre–Sobolev orthogonal polynomials: Asymptotic behavior and zeros

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ABSTRACT

We consider a varying discrete Sobolev inner product involving the Laguerre weight. Our aim is to study the asymptotic properties of the corresponding orthogonal polynomials and of their zeros. We are interested in Mehler–Heine type formulas because they describe the asymptotic differences between these Sobolev orthogonal polynomials and the classical Laguerre polynomials. Moreover, they give us an approximation of the zeros of the Sobolev polynomials in terms of the zeros of other special functions. We generalize some results appeared very recently in the literature for both the varying and non-varying cases.

1. Introduction

In this paper we deal with sequences of polynomials orthogonal with respect to a varying Sobolev inner product involving the Laguerre weight $w(x) = x^\alpha e^{-x}, \alpha > -1$, on the real nonnegative semiaxis $[0, +\infty)$. More precisely, we consider the inner product

$$(f,g)_n = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(x)g(x)x^\alpha e^{-x}dx + M_n f^{(j)}(0)g^{(j)}(0), \quad j \geqslant 0, \tag{1}$$

with $\alpha > -1$ and where $\{M_n\}_n$ is a sequence of nonnegative numbers satisfying

$$\lim_{n\to\infty} M_n n^\beta = M > 0, \quad \text{with } \beta \in \mathbb{R}. \tag{2}$$

This inner product generalizes one considered in [5,7], i.e., for $\alpha > -1$,

$$(f,g) = \int_0^\infty f(x)g(x)x^\alpha e^{-x}dx + Nf^{(j)}(0)g^{(j)}(0), \quad j, N \geqslant 0. \tag{3}$$

Thus, we will recover the results appearing in those papers when $\{M_n\}_n$ is a constant sequence. Note that for $M_n = N/\Gamma(\alpha+1)$,for all n, we have

$$\Gamma(\alpha+1)(f,g)_n = (f,g)$$

and it is necessary to take this into account for the technical details.

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Moreover, we want to give a qualitative interpretation of the asymptotic behavior of the orthogonal polynomials with respect to (1) in this general case. In such a sense, we prove that the *size* of the sequence $\{M_n\}_n$ has an influence on the asymptotic behavior of the orthogonal polynomials with respect to (1), but this influence is only local, that is, around the point where we have introduced the perturbation. In our case, this point is located at the origin. Thus, denoting by $L_n^{(\alpha)}(x)$ the classical Laguerre polynomials and by $L_n^{(\alpha,M_n)}(x)$ the orthogonal polynomials with respect to (1), first we will prove that

$$\lim_{n\to\infty}\frac{L_n^{(\alpha,M_n)}(x)}{L_n^{(\alpha)}(x)}=1, \tag{4}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,\infty)$. When $M_n=M$ for all n, (4) was already observed by several authors (see, for example, [1]). Then, we focus our attention on the local asymptotic behavior to find the differences between both sequences of orthogonal polynomials. In fact, we focus our attention on the limit behavior of the ratio

$$\frac{L_n^{(\alpha,M_n)}(x/n)}{n^\alpha},\quad \text{when } n\to\infty$$

and we will describe how the size of the sequence $\{M_n\}_n$ influences on the local asymptotics, i.e., essentially we have three possible cases: one of them is when the size of $\{M_n\}_n$ is negligible and therefore the Mehler–Heine type asymptotics for $\{L_n^{(\alpha,M_n)}\}_n$ and $\{L_n^{(\alpha)}\}_n$ are the same; another one is when the size of $\{M_n\}_n$ influences on the asymptotics; and in the third one we will prove that it is a convex combination of the two other cases. Thus, we generalize the results obtained in [3,4] for particular cases.

We also analyze the zeros of the polynomials $L_n^{(\alpha,M_n)}(x)$ and their asymptotic behavior as a consequence of the Mehler–Heine type formula.

According to our objectives, the structure of the paper is the following. In Section 2, we introduce the varying Laguerre–Sobolev type orthogonal polynomials and their basic properties. In Section 3, we provide our main results about the asymptotics of the polynomials $L_n^{(\alpha,M_n)}(x)$. Finally, Section 4 is devoted to the zeros of $L_n^{(\alpha,M_n)}(x)$, as well as we show some numerical computations for illustrating the results previously obtained.

2. Laguerre–Sobolev type orthogonal polynomials: the varying case

We consider the nonstandard and varying inner product

$$(f,g)_n=\frac{1}{\Gamma(\alpha+1)}\int_0^\infty f(x)g(x)x^\alpha e^{-x}dx+M_nf^{(j)}(0)g^{(j)}(0),\quad j\geqslant 0, \tag{5}$$

with $\alpha>-1$, and where $\{M_n\}_n$ is a sequence of nonnegative numbers such that

$$\lim_{n\to\infty}M_nn^\beta=M>0,\quad \beta\in\mathbb{R}. \tag{6}$$

This inner product is nonstandard because $(xf,g)_n\neq(f,xg)_n$, and thus the nice properties (three–term recurrence relation, Christoffel–Darboux formula, etc.) that we can deduce for standard orthogonal polynomials do not hold for the orthogonal polynomials with respect to (5). We should pay attention to the following fact: denoting by $L_n^{(\alpha,M_n)}(x)$ the orthogonal polynomials with respect to (5), then $(L_n^{(\alpha,M_n)},x^i)_n=0$, for $i=0,\dots,n-1$, but $(L_n^{(\alpha,M_n)},x^i)_{n-1}$ may be different from zero. In fact, for a sequence $\{M_n\}_n$ we have a sequence of orthogonal polynomials for each n, so we have a square tableau $\{L_k^{(\alpha,M_n)}\}_k$. Here, we treat with the diagonal of this tableau, i.e. $\{L_n^{(\alpha,M_n)}\}_n=\{L_0^{(\alpha,M_0)},L_1^{(\alpha,M_1)},\dots,L_i^{(\alpha,M_i)},\dots\}$.

When $M_n=0$ for all n, the inner product (5) becomes the Laguerre inner product whose orthogonal polynomials are denoted by $L_n^{(\alpha)}(x)$. We choose the same normalization for both sequences of orthogonal polynomials $\{L_n^{(\alpha,M_n)}\}_n$ and $\{L_n^{(\alpha)}(x)\}_n$.The leading coefficient of the polynomial of degree n in each family is $(-1)^n/n!$.

It is easy to observe that $L_n^{(\alpha,M_n)}(x)=L_n^{(\alpha)}(x)$ for $n=0,\dots,j-1$. A first step to get asymptotic properties is to obtain an adequate expression of the polynomials $L_n^{(\alpha,M_n)}(x)$ in terms of the classical Laguerre polynomials, i.e., to solve the connection problem. When $\{M_n\}_n$ is a constant sequence, this problem was solved in [6] where the author introduced Sobolev type orthogonal polynomials involving more derivatives. Very recently, in [7] the authors have given the explicit expression of those coefficients. Now, we rewrite Theorem 1 in [7] for the varying case.

Proposition 1. *We assume $L_{-1}^{(\alpha)}(x)\equiv 0$,and $\alpha>-1$. We have, for every $n\geqslant j$,*

$$L_n^{(\alpha,M_n)}(x)=L_n^{(\alpha)}(x)+\sum_{k=1}^{j+1}B_{n,k}^{[j]}L_{n-k}^{(\alpha+k)}(x),$$

where $B_{n,k}^{[j]}=\frac{A_{n,k}^{[j]}}{A_{n,0}^{[j]}}$,with