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A quasi-reversibility regularization method for an inverse heat conduction problem without initial data ☆

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ABSTRACT

In this paper, we study an inverse heat conduction problem without initial data in a bounded domain in which the Cauchy data at $x = 0$ are given and the solution in $0 < x \leqslant 1$ is sought. The problem is ill-posed in the sense that the solution (if it exists) does not depend continuously on the Cauchy data. In order to obtain a stable numerical solution, we propose a quasi-reversibility regularization method to solve the inverse heat conduction problem. Convergence estimates for the regularized solution are given. For numerical implementation, we provide a method of lines to deal with the regularized problem. Some numerical examples illustrate that the proposed method is reasonable and feasible.

1. Introduction

In many industrial applications one wants to determine the interior temperature or the surface temperature of a body where the interior or surface itself is inaccessible. In this case, we need to determine the interior or surface temperature from a measured temperature history at a fixed location inside the body or on the accessible boundary. In this paper, we consider an inverse heat conduction problem without initial data in which the temperature and heat flux are specified at $x = 0$ and the solution in $0 < x \leqslant 1$ is sought. As we know, this kind of the problem is severely ill-posed in Hadamard's sense [3,6,12,10,19]. That is, any small errors in Cauchy data will cause large errors in the solution. The classical numerical method, e.g., the finite difference method, would fail to give any reliable results if one tries to solve directly this problem. Hence, a special regularization method is necessary for stabilizing computations [7,9,11,13]. In the past years, many regularization methods have been developed for solving a related sideways heat equation problem such as the Tikhonov method [1], wavelet and wavelet-Galerkin method [6,15,14], wavelet and spectral regularization methods [8], a spectral regularization method [20], optimal approximations [17] and optimal filtering method [16], etc.

We know that there are many works for solving inverse heat conduction problems with known initial data. However, in many situations we can not know the initial condition because the heat process has already started before we estimate the problem [18]. In this paper, we propose a quasi-reversibility regularization method for solving an inverse conduction problem without initial data. A similar regularization method was used in [2] where the original heat equation was transformed into a hyperbolic equation and then got a well-posed problem on partial differential equations. The convergence results were given, but no error estimate was obtained. The sideways heat equation problem in a quarter plane was also solved by the quasi-reversibility regularization method, for instants, Eldén [3,4] where the author used the Fourier transform to get the

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exact solutions for the sideways heat problem and the modified problem. In this paper, we use a forth-order modified method same with one in [3] to obtain a regularized solution. The convergence estimates are given based on the Fourier series. For numerical implementation, we also propose a method of lines to solve the well-posed problem and then obtain a stable approximate solution.

The outline of the paper is as follows. In Section 2, the formulation of the problem and a quasi-reversibility regularization method are given. Section 3 gives the convergence estimates for the regularized solution. The method of lines is applied to obtain an approximate solution in Section 4. Several numerical examples are presented in Section 5 to illustrate the efficiency of the proposed method. Finally, we give some concluding remarks in Section 6.

2. Formulation of the problem and a quasi-reversibility regularization method

Consider an inverse heat conduction problem

$$\begin{cases} w_t = w_{xx}, & 0 < x < 1,\ 0 < t < 2\pi, \\ w(0,t) = f(t), & 0 \leqslant t \leqslant 2\pi, \\ w_x(0,t) = q(t), & 0 \leqslant t \leqslant 2\pi. \end{cases} \tag{2.1}$$

We want to determine the solution $w(x,t)$ for $0 < x \leqslant 1$ from the Cauchy data $f(t)$ and $q(t)$. Physically, we only obtain the measured Cauchy data with measurement errors. Suppose f^δ, $q^\delta \in L^2[0,2\pi]$ are measured data and satisfy

$$\|f - f^\delta\| \leqslant \delta, \quad \|q - q^\delta\| \leqslant \delta, \tag{2.2}$$

where $\|\cdot\|$ denotes the L^2-norm and the constant $\delta > 0$ represents a noisy level.

We can get a formal solution for problem (2.1), refer to [2],

$$w(x,t) = \sum_{n=-\infty}^{+\infty}\left[C_n e^{int}\cosh(\sqrt{in}x) + \frac{D_n}{\sqrt{in}}e^{int}\sinh(\sqrt{in}x)\right] \tag{2.3}$$

where

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt, \quad D_n = \frac{1}{2\pi}\int_0^{2\pi} q(t)e^{-int}dt \tag{2.4}$$

and

$$\sqrt{in} = \sqrt{\frac{|n|}{2}}(1+\sigma i), \quad \sigma = sign(n), \quad n \in \mathbb{Z}. \tag{2.5}$$

Remark 2.1. In (2.3), for $n = 0$, $\frac{\sinh(\sqrt{in}x)}{\sqrt{in}}$ is defined as x, since $\lim_{n\to 0}\frac{\sinh(\sqrt{in}x)}{\sqrt{in}} = x$.

In the following, we split the Cauchy problem (2.1) into two independent Cauchy problems. Let $u(x,t)$ and $v(x,t)$ be the solution of the following problem, respectively,

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1,\ 0 < t < 2\pi, \\ u(0,t) = f(t), & 0 \leqslant t \leqslant 2\pi, \\ u_x(0,t) = 0, & 0 \leqslant t \leqslant 2\pi \end{cases} \tag{2.6}$$

and

$$\begin{cases} v_t = v_{xx}, & 0 < x < 1,\ 0 < t < 2\pi, \\ v(0,t) = 0, & 0 \leqslant t \leqslant 2\pi, \\ v_x(0,t) = q(t), & 0 \leqslant t \leqslant 2\pi, \end{cases} \tag{2.7}$$

then $w = u + v$ is the solution of problem (2.1). Therefore, we only need solve problems (2.6) and (2.7), respectively. Since Cauchy problems (2.6) and (2.7) are all severely ill-posed, we should apply a regularization method to solve them.

The exact solution of problem (2.6) has the following formal solution, refer to [2],

$$u(x,t) = \sum_{n=-\infty}^{+\infty} C_n e^{int}\cosh(\sqrt{in}x) \tag{2.8}$$

where C_n is given by (2.4). Note that the real part of $\cosh(\sqrt{in}x)$ is positive, the small error in the Dirichlet data will be amplified by the factor $\Re(\cosh(\sqrt{in}x))$ for $0 < x \leqslant 1$, so the Cauchy problem (2.6) is severely ill-posed. We must use some regularization methods to deal with this problem. In this paper, we apply a quasi-reversibility method to construct an approximate solution for problem (2.6). That is to find the solution of the following problem