Applied Mathematics and Computation 219 (2013) 10974–10982

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On numerical approximations of the area of the generalized Mandelbrot sets

Ioannis Andreadis [a], Theodoros E. Karakasidis [b,*]

[a] International School of The Hague, Wijndaelerduin 1, 2554 BX The Hague, The Netherlands
[b] Department of Civil Engineering, University of Thessaly, GR-38334 Volos, Greece

ARTICLE INFO

Keywords:
Generalized Mandelbrot set
Area
Finite escape algorithm
Lattice points

ABSTRACT

In the present work, the area of the generalized Mandelbrot sets is defined as the double limit of the areas of the plotted generalized Mandelbrot sets in a given square lattice, using the finite escape algorithm, while the lattice resolution and the number of iteration counts, used to plot them, tends to infinity. The asymptotic behavior of the areas of the generalized Mandelbrot sets in terms of their degree growth is investigated. Finally, numerical approximations of the area of the Mandelbrot set are proposed by using tools from regression analysis.

1. Introduction

In this work, we propose a mathematical framework for defining the area of the generalized Mandelbrot sets [1–4] based on the calculation of the areas of the numerically plotted generalized Mandelbrot sets on a square lattice using the finite escape algorithm [5].

Firstly, we consider a square lattice of N^2 points, where N denotes the lattice resolution [6]; the number of points on the side of the square lattice. Then, we define as a projected generalized Mandelbrot set in a square lattice, the plotted image of the generalized Mandelbrot set on the given lattice associated to a maximum number of iteration counts.

Then, the area of the projected generalized Mandelbrot set is calculated based on the counting pixel method as it was proposed in the cased of the area of the Mandelbrot set in [6]. Finally, we define the area of the generalized Mandelbrot set as the double limit of the area of the projected generalized Mandelbrot sets while the resolution and the maximum number of iteration counts used to plot them tend to infinity.

In the case of the Mandelbrot set, this approach was suggested by Rabenhorst in [6] who conjectured that a value of the area of the Mandelbrot set to be 1.508 (to 4 significant digits). Subsequently, Ewing and Schober [7,8], using analytical methods estimated that the area of the Mandelbrot set is 1.7274 (to 5 significant digits). Fischer and Hill [9,10] combined both analytical and numerical methods; they found an estimation of 1.5613 (to 5 significant digits). (For further details, there is the web-page maintained by Muffano [11] dedicated to the calculations of the area of the Mandelbrot set).

Then, we presented numerical plots of the generalized projected Mandelbrot sets for various degrees and we calculate the asymptotic behavior of their areas in terms of their degree growth. The numerical results obtained are in agreement with the theoretical result obtained by Boyd and Schulz [4] when they proved that the limit of the generalized Mandelbrot set as their degree goes to the infinity is the unit disk.

The current paper is divided in four parts. In the first part, we describe briefly the finite escape algorithm and how it is applied to plot a projected generalized Mandelbrot set in a square lattice with a given resolution and a given maximum

* Corresponding author.
E-mail addresses: i.andreadis@ish-rijnlandslyceum.nl (I. Andreadis), thkarak@uth.gr (T.E. Karakasidis).

0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2013.04.052

number of iteration counts. Then, we present the application of the pixel counting method to calculate the areas of the projected generalized Mandelbrot sets as it was proposed in [6] for the case of the area of the Mandelbrot set.

In the second part we present numerical plotting of various generalized projected Mandelbrot sets and we calculate a model of the asymptotic behavior of their areas in terms of their degree growth.

In the third part we define the area of the generalized Mandelbrot sets as the double limit [12,13] of the area of the projected generalized Mandelbrot sets plotted in a square lattice as the maximum number of iteration counts and the lattice resolution, used to plot them, tend to infinity. Afterwards, using the Cauchy convergence criterion [13], we prove the existence of such a limit.

Thereafter, in the fourth part, we focus in the case of the projected Mandelbrot set and we calculate various areas of the projected Mandelbrot sets while varying the maximum number of iteration counts and the resolution of the square lattice used to plot them. Then, we construct a 3-D regression model for modelling those values. Thereafter, by using a Gaussian fit [14] we calculate an approximation value of 1.5101 (to 5 significant digits) for the area of the Mandelbrot set. Then, we consider a nested subsequence of square lattices; one contained in the subsequent one. Afterwards, while fixing the maximum number of iteration counts, we construct a subsequence of the double sequence of the areas of the projected Mandelbrot sets, by increasing the resolution of those lattices. Then, we provide a condition for the existence of the limit of that subsequence by using an assumption of asymptotically equivalent function [15]. Finally, using tools of regression analysis, we provide numerical values of that limit and hence numerical approximations of the area of the Mandelbrot set.

2. On the finite escape algorithm used to plot the projected generalized Mandelbrot sets

Let us recall the definition of the iteration process defined by a complex map $Q_{a,c}$ of degree α a real number, as:

$$Q_{\alpha,c}(z) = z^{\alpha} + c, \tag{1}$$

with c, z complex numbers.

Let us recall the definition of the generalized Mandelbrot set of the complex quadratic map $Q_{\alpha,c}$ which we denote by $M(Q_{\alpha,c})$. We fix the origin (0,0), and consider different values of the parameters $C = (c_1,c_2)$. The generalized Mandelbrot set of the map $Q_{\alpha,c}$, is the set of all the values of parameters (c_1,c_2) such that $\lim_{n\to\infty}|Q_{\alpha,c}{}^{(n)}(0,0)| < \infty$, where $Q^{(n)}_{\alpha,c}$ denoted the nth iteration of the map $Q_{\alpha,c}$.

In the following, we recall briefly the escape-time algorithm as explained in [5], for the case of the Mandelbrot set, as it is applied for the numerical calculation of the generalized Mandelbrot set. Initially, we fix an interval of the space of parameters (c_1,c_2) as follows $-2 \leqslant c_1 \leqslant 2$ and $-2 \leqslant c_2 \leqslant 2$, as those considered in [6]. Subsequently, we consider a lattice of parameters values with various numbers of points. Then, we set the maximum number of iteration counts to 500 and we calculate the value of the distance r from the origin up to the 500 iterations counts. If $r \leqslant 10$, then we keep the point (x_o,y_o) in a file, otherwise we ignore this point and, finally, we plot the resulting file, which gives us the corresponding projected generalized Mandelbrot set in the given lattice of points. We define also that this point (c_1,c_2) bares the generalized Mandelbrot Property [16].

In Fig. 1 we present the projected Mandelbrot sets, $\alpha = 2$ on a square lattice with a maximum number of iteration counts equal to 500 and with a lattice resolution (a) 500, and (b) 1000 respectively.

In Fig. 2 we present the projected generalized Mandelbrot sets, on a square lattice with a maximum number of iteration counts equal to 500 and with a lattice resolution 500 for $\alpha = 10, 30, 50, 70, 90$ and 100.

The numerical results indicate that at the limit of the degree growth to the infinity the generalized Mandelbrot set approaches the unit disk as it theoretically proved by Boyd and Schulz [4].

3. On the pixel counting method to calculate the area of a projected generalized Mandelbrot set

Let us now extend the counting pixel method used by Rabenhorst in 1987 [6] to evaluate the area of the Mandelbrot set for the case of the area of the generalized projected Mandelbrot sets based on the finite escape algorithm [5].

Firstly we consider a square lattice of points where the projected generalized Mandelbrot set is plotted, based on the calculations presented in [16]. For any four real numbers a, b, c and d such that $a < b$ and $c < d$, we consider a lattice of points (x,y) with x an element of the interval $[a,b]$ and y an element of the interval $[c,d]$. Then, for any lattice resolution N, we construct N subintervals of the x-interval $[a,b]$, defined via the formula: $a + (i-1)\left(\frac{b-a}{N-1}\right)$, with $1 \leqslant i \leqslant N$ and N subintervals of the y-interval $[c,d]$, defined via the formula: $c + (j-1)\left(\frac{d-c}{N-1}\right)$, with $1 \leqslant j \leqslant N$. Hence, we obtain N^2 points on the lattice with coordinates $P\left(a + (i-1)\left(\frac{b-a}{N-1}\right), c + (j-1)\left(\frac{d-c}{N-1}\right)\right)$, with $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant N$.

Thereafter, we denote by $M_{S;N}(Q_{\alpha,C})$ the projected Mandelbrot set with S the maximum number of iteration counts and N the lattice resolution. Then we define the number of generalized Mandelbrot points n $(M_{S;N}(Q_{\alpha,C})$ is the counting of all the points of the square lattice that belongs to $M_{S;N}(Q_{\alpha,C})$. Then, we denote the area of the $M_{S;N}(Q_{\alpha,C})$, based on the counting pixel method, as $\mathbf{A}^{\alpha}(S,N)$ which is defined as the ratio of the number of the generalized Mandelbrot points over all the total number of the square lattice point times the area of the lattice: