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Improved ninth order WENO scheme for hyperbolic conservation laws

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ABSTRACT

In this paper, we construct a new ninth order WENO scheme by extending the improved fifth order WENO introduced in [R. Borges, M. Carmona, B. Costa, W. Sun Don, J. Comput. Phys. 227 (2008) 3191–3211] (IWENO). We use the central-upwind flux [A. Kurganov, S. Noelle, G. Petrova, SIAM J. Sci. Comp. 23 (2001) 707–740] which is simple, universal and efficient. The numerical solution is advanced in time by the ninth order linear strong-stability-preserving Runge–Kutta (ℓSSPRK) scheme. The resulting scheme improves the convergence and accuracy at critical points of smooth parts of solution as well as decrease the dissipation near discontinuities. This is especially for long time evolution problems. Numerical experiments of the new scheme for one and two dimensional problems are reported. The results demonstrate that the proposed scheme is superior to the original IWENO and classical WENO schemes.

1. Introduction

Hyperbolic conservation laws describe a wide variety of phenomena in different areas of physics and other disciplines. Analytical solutions are available only in a few model problems and one thus has to rely on numerical methods for solving problems of practical interest. The solutions of hyperbolic equations laws may develop discontinuities even when the initial condition is smooth. Therefore the numerical method should compute such discontinuities with the correct position and without spurious oscillations and yet achieve high order of accuracy in the smooth regions. Amongst successful development in this direction we have TVD schemes [4], ENO schemes [5], WENO methods [7] and more recently, ADER methods [15].

TVD schemes are second order (or higher) accurate and avoid oscillations by locally reverting to first order accuracy near discontinuities and extrema. Therefore TVD schemes are unsuitable for special applications areas such as acoustics, compressible turbulence and problems involving long time evolution wave propagation. In these applications extrema are clipped as time evolves and numerical dissipation may become dominant.

In [15] ADER approach is recently developed as a generalization of the upwind Godunov's method and rely on the solution of a generalized Riemann problem (GRP) with initial condition consisting of polynomial functions of arbitrary order. The solution of this difficult problem is reduced to the solution of a sequence of m conventional Riemann problems (RP). For non-linear problems the first of these is a nonlinear RP and the remaining ones are linear RPs for the k-th order spatial derivatives of the initial conditions, with $k = 0, 1, \ldots m - 1$, where m is arbitrary and is the order of accuracy of the resulting scheme. The ADER schemes are conservative, one step, explicit and fully discrete, requiring only the computation of the intercell flux to advance the solution by a full time step. To evaluate the numerical flux, in the ADER schemes, one solves the generalized RP,

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exactly or numerically with initial condition consisting of two arbitrary but smooth functions using a semi-analytical method. To avoid the Gibbs phenomenon the ADER approach relies on WENO reconstructions of the data from cell averages. However, the superior accuracy of the ADER upwind schemes comes at a cost, one must solve exactly or approximately, the RP. Conventional Riemann solvers are usually complex and are not available for many hyperbolic systems of practical interest.

Weighted essentially non-oscillatory (WENO) schemes [1,7,10]. WENO schemes are based upon the essentially non-oscillatory (ENO) schemes [5,14,11].

The key idea in the r-th order ENO reconstruction procedure is to r possible stencil covering the given cell and select only one, the smoothest, stencil. The reconstructions polynomial is then built using this selected stencil. The WENO reconstruction [7] takes a convex combination of all r stencils with nonlinear solution adaptive weights. The design of the weights involves local estimates of the smoothness of the solution in each possible stencil so that the reconstruction achieves $(2r-1)$-th order of spatial accuracy in smooth regions and emulates the r-th order ENO reconstruction near discontinuities.

In [6], Henrick et al. studied the classical WENO scheme [7] and discovered that the classical choice of the smoothness indicators in [7] generated weights failed to recover the fifth order of the scheme at the critical points where the first or higher derivatives of the solution vanish. Also, in [16,17] necessary and sufficient conditions on the weights, for fifth order, were derived and a corrected mapping to be applied to the classical weights was devised. The resulting mapped WENO scheme of [6] recovered the fifth order of convergence at critical points of a smooth function and presented sharper results near discontinuities.

In [2], another technique was presented to improve on the classical smoothness indicators to build weights providing a new WENO reconstruction (IWENO). The use of these weights improves the convergence order at the critical points of a smooth function, as well as decrease the dissipation near discontinuities. The computational cost of the improved scheme [2] is the same as the classical WENO [7] and around 75% of the mapped WENO scheme [6].

Usually, only lower order (first order) fluxes are used as the building block for the construction of higher order schemes.

The purpose of this paper is three fold. Firstly, we construct a new ninth order WENO interpolation procedure. The interpolation results from extending the improved fifth order WENO scheme [2]. Secondly, we propose to use a more general central-upwind scheme introduced in [8,9] which is simple, universal and efficient scheme. Thirdly, for the time integration we use the ninth order linear strong-stability-preserving Runge–Kutta (ℓSSPRK) [3]. The resulting scheme improves the convergence and accuracy at critical points of smooth parts of solution as well as decrease the dissipation near discontinuities. This is especially so for long time evolution problems. The main advantages of the new scheme are: it is more accurate and can be used for problems with non-convex fluxes.

The paper is organized as follow. In Section 2 we briefly review the WENO reconstruction. In Section 3 we introduce our new ninth order WENO reconstruction. In Section 4 we review the derivation of the central-upwind flux [8,9]. The linear strong-stability-preserving Runge–Kutta method (ℓSSPRK) [3] of time discretization is briefly reviewed in Section 5. Section 6 presents the results of number of numerical tests of our method. Extension of the scheme for systems equations in two dimensions is described in Section 7.

2. Numerical methods

In this section we review the components we use to construct our fourth order central-upwind scheme for hyperbolic systems in conservative form

$$u_t + [f(u)]_x = 0, \tag{2.1}$$

along with initial and boundary conditions. Here $u(x,t)$ is the vector of unknown conservative variables and $f(u)$ is the physical flux vector. Throughout this paper, we consider only uniform grids and use the following notation- let $x_j = j\Delta x,\ x_{j\pm\frac{1}{2}} = x_j \pm \frac{1}{2}\Delta x,\ t^n = n\Delta t,\ u_j^n = u(x_j, t^n)$ and the cell $I_j = \left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right]$, where Δx and Δt are small spatial and time scales. Consider a control volume in x-space $\left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right]$. Integrating (2.1) with respect to x over the volume and keeping the time variable continuous we obtain the semi-discrete finite volume scheme which is in fact a system of ordinary differential equations (ODEs)

$$\frac{d}{dt}u_j(t) = -\frac{1}{\Delta x}\left\{F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right\} = L_j(u), \tag{2.2}$$

where $u_j(t)$ is the space average of the solution in the cell I_j at time t and $F_{j+\frac{1}{2}}$ is the numerical flux at $x = x_{j+\frac{1}{2}}$ and time t

$$u_j(t) = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t)dx, \quad F_{j+\frac{1}{2}} = F(u_{j+\frac{1}{2}}(t)). \tag{2.3}$$

The steps to follow in the implementation of the numerical scheme (2.2) can be described as follows:

(i) The first step in the derivation of the approximate solution is to generate a piecewise polynomial reconstruction from the cell averages. Such a global reconstruction is defined as

$$u(x,t^n) = P_j^n(x), \quad x_{j-\frac{1}{2}} \le x \le x_{j+\frac{1}{2}}, \tag{2.4}$$

where $P_j^n(x)$ is polynomial of a suitable degree. In each cell I_j the reconstruction should be conservative, formally r-th order accurate and non-oscillatory. As a result, at each cell interface $x_{j+\frac{1}{2}}$ between cells j and $j+1$ the reconstruction produces two