Applied Mathematics and Computation 219 (2013) 8407–8412

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On a symmetric system of max-type difference equations

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ARTICLE INFO

Key words:
Max-type system
Difference equation
Positive solution
Boundedness
Global attractivity

ABSTRACT

We study behavior of positive solutions of the following max-type system of difference equations

$$x_{n+1} = \max\left\{c, \frac{y_n^p}{x_{n-1}^p}\right\}, \quad y_{n+1} = \max\left\{c, \frac{x_n^p}{y_{n-1}^p}\right\}, \quad n \in \mathbb{N}_0,$$

where $p, c \in (0, \infty)$, extending some results in the literature. Among other results, we prove that if $p, c \in (0, 1)$, then every positive solution of the system converges to $(1, 1)$.

1. Introduction

Max-type difference equations, which appeared for the first time in control theory, have attracted some attention recently (see, e.g., [1–8,16,19–25,27,30,31,37–39] and the related references therein). However, although there is a considerable interest in studying systems of difference equations (see, e.g., [9–17,26,28,29,31–36] and the related references therein), only a few papers deal with systems of max-type difference equations (see, e.g. [16,30,31]).

In [2] we gave an elegant proof, which is related to the proof of Theorem 2 in [18], of the result which says that every positive solution of the next difference equation

$$x_{n+1} = \max\left\{c, \frac{x_n}{x_{n-(k+1)}}\right\}, \quad n \in \mathbb{N}_0, \tag{1}$$

where parameter c is positive, is bounded.

This result motivated us to investigate the behavior of positive solutions of the following max-type difference equation

$$x_{n+1} = \max\left\{c, \frac{x_n^p}{x_{n-1}^p}\right\}, \quad n \in \mathbb{N}_0, \tag{2}$$

where c and p are positive numbers (see [19]).

Here we study the following max-type system of difference equations

$$x_{n+1} = \max\left\{c, \frac{y_n^p}{x_{n-1}^p}\right\}, \quad y_{n+1} = \max\left\{c, \frac{x_n^p}{y_{n-1}^p}\right\}, \quad n \in \mathbb{N}_0, \tag{3}$$

where $p, c \in (0, \infty)$, which is a natural extension of Eq. (2).

We show that positive solutions of system (3) have properties similar to those of positive solutions of Eq. (2) (solution $(x_n, y_n)_{n \geqslant -1}$ of system (3) is called positive if $x_n > 0$ and $y_n > 0$ for every $n \geqslant -1$). More specifically, we prove the following: all positive solutions of system (3) are bounded when $p \in (0, 4)$; every positive solution is eventually equal to (c, c), when

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0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.02.008

$p \in (0,4)$ and $c \geqslant 1$; there are unbounded solutions for $p \geqslant 4$; and for $p, c \in (0,1)$, all positive solutions of system (3) converge to $(1,1)$.

2. Boundedness character of system (3)

This section is devoted to studying the boundedness character of positive solutions of system (3).

2.1. Case $p \in (0,4), c > 0$

In this case the following result holds.

Theorem 1. *Let $p \in (0,4)$ and $c > 0$. Then all positive solutions of system (3) are bounded.*

Proof. Assume that $(x_n, y_n)_{n \geqslant -1}$ is a positive solution of system (3). Then the following estimate obviously holds

$$\min\{x_n, y_n\} \geqslant c, \quad n \in \mathbb{N}. \tag{4}$$

Now we show that the sequence $(x_n)_{n \geqslant -1}$ is bounded. Since system (3) is symmetric with respect to variables x_n and y_n, then if we show that the sequence $(x_n)_{n \geqslant -1}$ is bounded, it will follow that the sequence $(y_n)_{n \geqslant -1}$ is bounded too, which will imply the boundedness of solution $(x_n, y_n)_{n \geqslant -1}$.

By using (3) we obtain

$$x_{n+1} = \max\left\{c, \frac{y_n^p}{x_{n-1}^p}\right\} = \max\left\{c, \left(\frac{y_n}{x_{n-1}}\right)^p\right\} = \max\left\{c, \left(\max\left\{\frac{c}{x_{n-1}}, \frac{x_{n-1}^{p-1}}{y_{n-2}^p}\right\}\right)^p\right\}. \tag{5}$$

If $p \in (0,1]$, then from (4) and (5) we have that

$$x_{n+1} = \max\left\{c, \left(\max\left\{\frac{c}{x_{n-1}}, \frac{1}{x_{n-1}^{1-p} y_{n-2}^p}\right\}\right)^p\right\} \leqslant \max\left\{c, 1, \frac{1}{c^p}\right\} \tag{6}$$

for $n \geqslant 3$. From (4) and (6) the boundedness of $(x_n)_{n \geqslant -1}$ follows in this case.

Now assume that $p > 1$. Then from (5) and (3), we have

$$x_{n+1} = \max\left\{c, \left(\max\left\{\frac{c}{x_{n-1}}, \left(\frac{x_{n-1}}{y_{n-2}^{\frac{p}{p-1}}}\right)^{p-1}\right\}\right)^p\right\} = \max\left\{c, \left(\max\left\{\frac{c}{x_{n-1}}, \left(\max\left\{\frac{c}{y_{n-2}^{\frac{p}{p-1}}}, \frac{y_{n-2}^{p-\frac{p}{p-1}}}{x_{n-3}^p}\right\}\right)^{p-1}\right\}\right)^p\right\}. \tag{7}$$

If $p \in (1,2]$, then from (4) and (7) we have that

$$x_{n+1} = \max\left\{c, \left(\max\left\{\frac{c}{x_{n-1}}, \left(\max\left\{\frac{c}{y_{n-2}^{\frac{p}{p-1}}}, \frac{1}{y_{n-2}^{\frac{p(2-p)}{p-1}} x_{n-3}^p}\right\}\right)^{p-1}\right\}\right)^p\right\} \leqslant \max\left\{c, 1, \frac{1}{c^p}, \frac{1}{c^{p^2}}\right\}, \quad \text{for} \quad n \geqslant 4. \tag{8}$$

From (4) and (8) the boundedness of $(x_n)_{n \geqslant -1}$ follows in this case.

Continuing with this procedure and using a simple inductive argument, we have that for each fixed $l \in \mathbb{N}$

$$\begin{aligned} x_{n+1} &= \max\left\{c, \left(\max\left\{\frac{c}{x_{n-1}}, \left(\max\left\{\frac{c}{y_{n-2}^{\frac{p}{p-1}}}, \left(\frac{y_{n-2}}{x_{n-3}^{p/(p-\frac{p}{p-1})}}\right)^{p-\frac{p}{p-1}}\right\}\right)^{p-1}\right\}\right)^p\right\} = \cdots \\ &= \max\left\{c, \left(\max\left\{\frac{c}{x_{n-1}}, \left(\cdots, \left(\max\left\{\frac{c}{x_{n-2l+1}^{p_{2l-1}}}, \frac{x_{n-2l+1}^{p-p_{2l-1}}}{y_{n-2l}^p}\right\}\right)^{p-p_{2l-2}} \cdots\right)^{p-1}\right\}\right)^p\right\} \end{aligned} \tag{9}$$

for $n \geqslant 2l+1$, and

$$\begin{aligned} x_{n+1} &= \max\left\{c, \left(\max\left\{\frac{c}{x_{n-1}}, \left(\max\left\{\frac{c}{y_{n-2}^{\frac{p}{p-1}}}, \left(\frac{y_{n-2}}{x_{n-3}^{p/(p-\frac{p}{p-1})}}\right)^{p-\frac{p}{p-1}}\right\}\right)^{p-1}\right\}\right)^p\right\} = \cdots \\ &= \max\left\{c, \left(\max\left\{\frac{c}{x_{n-1}}, \left(\cdots, \left(\max\left\{\frac{c}{y_{n-2l}^{p_{2l}}}, \frac{y_{n-2l}^{p-p_{2l}}}{x_{n-2l-1}^p}\right\}\right)^{p-p_{2l-1}} \cdots\right)^{p-1}\right\}\right)^p\right\} \end{aligned} \tag{10}$$