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# Newton Lavrentiev regularization for ill-posed operator equations in Hilbert scales

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ABSTRACT

In this paper we consider the two step method for approximately solving the ill-posed operator equation $F(x) = f$, where $F : D(F) \subseteq X \to X$, is a nonlinear monotone operator defined on a real Hilbert space $X$, in the setting of Hilbert scales. We derive the error estimates by selecting the regularization parameter $\alpha$ according to the adaptive method considered by Pereverzev and Schock in (2005), when the available data is $f^\delta$ with $\|f - f^\delta\| \leqslant \delta$. The error estimate obtained in the setting of Hilbert scales $\{X_r\}_{r\in R}$ generated by a densely defined, linear, unbounded, strictly positive self adjoint operator $L : D(L) \subset X \to X$ is of optimal order.



## 1. Introduction

This paper is devoted to the study of nonlinear ill-posed operator equation

$$F(x) = f, \tag{1.1}$$

where $F : D(F) \subset X \to X$ is a nonlinear monotone operator. Here, $D(F)$ is the domain of $F$ and $X$ is a real Hilbert space with corresponding inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$. Recall that ([19,20]) $F$ is a monotone operator if it satisfies the relation

$$\langle F(x) - F(y), x - y \rangle \geqslant 0, \quad \forall x, y \in D(F).$$

Throughout we assume that $f \in R(F)$, the range of $F$, so that (1.1) has a solution $\hat{x}$, but due to the nonlinearity of $F$ this solution need not be unique. Therefore we consider an $x_0$-minimal norm solution ($x_0$-MNS) of (1.1). Recall that (see, [1,11,18]) a solution $\hat{x} \in D(F)$ of (1.1) is called an $x_0$-MNS of (1.1), if $F(\hat{x}) = f$ and $\|\hat{x} - x_0\| = min\{\|x - x_0\| : F(x) = f, x \in D(F)\}$.

Throughout this paper we assume the existence of an $x_0$-MNS, $\hat{x}$ for exact data f, i.e.,

$$F(\hat{x}) = f$$

and the element $x_0$ is assumed to be known. Further we assume that $f^\delta \in X$ are the available noisy data with $\|f - f^\delta\| \leqslant \delta$.

Different methods have been considered by various authers for approximately solving nonlinear ill-posed operator equations, for example see, [1,3,4,19,20], etc. and the references therein.

Recently, in [8] George and Pareth considered a two-step method defined iterartively as;

$$y^\delta_{n,\alpha} = x^\delta_{n,\alpha} - R_\alpha(x_0)^{-1}[F(x^\delta_{n,\alpha}) - f^\delta + \alpha(x^\delta_{n,\alpha} - x_0)] \tag{1.2}$$

and

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$$x^{\delta}_{n+1,\alpha} = y^{\delta}_{n,\alpha} - R_{\alpha}(x_0)^{-1}[F(y^{\delta}_{n,\alpha}) - f^{\delta} + \alpha(y^{\delta}_{n,\alpha} - x_0)], \tag{1.3}$$

where $x^{\delta}_{0,\alpha} := x_0, R_{\alpha}(x_0) = F'(x_0) + \alpha I$ and $F'(x_0)$ is the Fréchet derivative of $F$ at $x_0$ for approximately solving (1.1).

In this paper we consider, the Hilbert scales variant of (1.2) and (1.3) for obtaining better convergence rates.

The paper is organized as: In Section 2, we give the preliminaries. The proposed method, the error bounds and parameter choice, the adaptive scheme and stopping rule are given in Section 3. Finally we conclude the paper in Section 4.

## 2. Preliminaries

Let $L : D(L) \subset X \to X$ be a densely defined unbounded self adjoint strictly positive operator. We consider a Hilbert scales $\{X_r\}_{r\in R}$ (see, [2,9,10,13–15]) induced by $L$, i.e., $X_r$ is the completion of $D := \cap_{k=0}^{\infty} D(L^k)$ with respect to the Hilbert space norm

$$\|x\|_r = \|L^r x\|, \quad r \in R.$$

Throughout this paper we will be using the following assumptions.

**Assumption 2.1.** There exist constants $a \geqslant 0, 0 < m \leqslant M < \infty$ such that

$$m\|h\|_{-a} \leqslant \|F'(x_0)h\| \leqslant M\|h\|_{-a}, \quad h \in X.$$

Note that the above assumption is weaker than the Assumption 3(a) in [10]. Let

$$A_s = L^{-\frac{s}{2}} F'(x_0) L^{-\frac{s}{2}},$$

$f(v) = min\{m^v, M^v\}$ and $g(v) = max\{m^v, M^v\}, \quad v \in R, \ |v| \leqslant 1.$

The following proposition is important for proving the results in this paper.

**Proposition 2.2** (See [5, Proposition 3.1]). *For $s \geqslant 0$ and $|v| \leqslant 1$,*

$$f\left(\frac{v}{2}\right)\|x\|_{-\frac{v}{2}(s+a)} \leqslant \|A_s^{v/2}x\| \leqslant g\left(\frac{v}{2}\right)\|x\|_{-\frac{v}{2}(s+a)}, \quad x \in X.$$

Using the above proposition, we prove the following Lemma, which is used extensively to prove the results of this paper.

**Lemma 2.3.** *Let Assumption 2.1 hold. Then for all $h \in X$,*

$$\|(F'(x_0) + \alpha L^s)^{-1}F'(x_0)h\| \leqslant \psi(s)\|h\|, \text{ where } \psi(s) = \frac{g\left(\frac{s}{2(s+a)}\right)}{f\left(\frac{s}{2(s+a)}\right)}.$$

**Proof.** Note that,

$$\|(F'(x_0) + \alpha L^s)^{-1}F'(x_0)h\| = \|L^{\frac{-s}{2}}(A_s + \alpha I)^{-1}A_s L^{\frac{s}{2}}h\| \leqslant \frac{1}{f\left(\frac{s}{2(s+a)}\right)}\left\|A_s^{\frac{s}{2(s+a)}}(A_s + \alpha I)^{-1}A_s L^{\frac{s}{2}}h\right\|$$

$$\leqslant \frac{1}{f\left(\frac{s}{2(s+a)}\right)}\|(A_s + \alpha I)^{-1}A_s\|\left\|A_s^{\frac{s}{2(s+a)}}L^{\frac{s}{2}}h\right\| \leqslant \psi(s)\|h\|.$$

The last step follows from the spectral properties of the self adjoint operator $A_s, s > 0$. □

## 3. Newton Lavrentiev method in Hilbert scales

In this section we consider the Hilbert scales variant of the method (1.2) and (1.3). Define

$$y^{\delta}_{n,\alpha,s} = x^{\delta}_{n,\alpha,s} - (F'(x_0) + \alpha L^s)^{-1}[F(x^{\delta}_{n,\alpha,s}) - f^{\delta} + \alpha L^s(x^{\delta}_{n,\alpha,s} - x_0)] \tag{3.4}$$

and

$$x^{\delta}_{n+1,\alpha,s} = y^{\delta}_{n,\alpha,s} - (F'(x_0) + \alpha L^s)^{-1}[F(y^{\delta}_{n,\alpha,s}) - f^{\delta} + \alpha L^s(y^{\delta}_{n,\alpha,s} - x_0)], \tag{3.5}$$

where $x^{\delta}_{0,\alpha,s} := x_0$, is the initial approximation for the solution $\hat{x}$ of (1.1). We will be selecting the regularization parameter $\alpha = \alpha_i$ from some finite set

$$D_N = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N\},$$

using the adaptive method considered by Perverzev and Schock in [16].