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Resolvents and solutions of singular Volterra integral equations with separable kernels

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In honor of Professor T. A. Burton for his seminal contributions to Liapunov theory for integral and fractional differential equations

ABSTRACT

The Volterra integral equation

$$x(t) = a(t) + \int_0^t B(t,s)x(s)\,ds \tag{1}$$

with a kernel of the form $B(t,s) = p(t)q(s)$ is investigated, where a, p, and q are functions that are defined a.e. on an interval $[0,T]$ and are measurable. The main result of this paper states that if qa is Lebesgue integrable on $[0,T]$, the sign of $B(t,s)$ does not change for almost all (t,s), and if there is a function f that is continuous on $[0,T]$, except possibly at countably many points, with $B(t,t) = f(t)$ a.e. on $[0,T]$, then the function x defined by

$$x(t) := a(t) + \int_0^t R(t,s)a(s)\,ds, \tag{2}$$

where

$$R(t,s) := B(t,s)e^{\int_s^t B(u,u)\,du}, \tag{3}$$

solves (1) a.e. on $[0,T]$. Three diverse examples illustrate the efficacy of using (2) and (3) to calculate solutions of (1). Two of the examples involve singular kernels: the solution of one of them is nowhere continuous on $(0,T)$.

1. Introduction

Volterra integral equations of the form

$$x(t) = a(t) + \int_0^t B(t,s)x(s)\,ds \tag{1.1}$$

with separable kernels crop up in certain applications, such as in some heat conduction problems with mixed-type boundary conditions. Lima and Diogo point this out in a paper [12, p. 538] in which they investigate the numerical solutions of such equations. The purpose of this paper is to show that for a kernel $B(t,s)$ with the separable form $p(t)q(s)$ there is a closed-form formula for the resolvent of the kernel, even if it has singularities, in terms of which the solution of (1.1) can be expressed. Furthermore, closed-form solutions of (1.1) can be calculated for a variety of specific equations as we shall demonstrate in Section 4.

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Peter Linz, in his monograph on Volterra equations [13], states and proves a result for the case of continuous a, p, and q from which solutions of (1.1) can be calculated [13, pp. 7–10]. We follow his presentation up to a point but then modify it to obtain Theorem 1.1 below. Its formulation, particularly (1.3), is scarcely known, if not unknown.

We begin by first supposing that a, p, and q are differentiable on $[0, T]$ and that $p(t) \neq 0$. Dividing (1.1) by $p(t)$ and letting $y(t) := x(t)/p(t)$, we get

$$y(t) = \frac{a(t)}{p(t)} + \int_0^t p(s)q(s)y(s)\,ds.$$

Differentiation then yields the linear differential equation

$$y'(t) - p(t)q(t)y(t) = \frac{d}{dt}\left[\frac{a(t)}{p(t)}\right].$$

Multiplying by the integrating factor

$$\mu(t) := e^{-\int_0^t p(s)q(s)\,ds},$$

we obtain

$$\frac{d}{dt}[\mu(t)y(t)] = \mu(t)\frac{d}{dt}\left[\frac{a(t)}{p(t)}\right],$$

which can also be written as

$$\frac{d}{dt}\left[\mu(t)y(t) - \mu(t)\frac{a(t)}{p(t)}\right] = -\frac{a(t)}{p(t)}\mu'(t).$$

Integration yields

$$\mu(t)y(t) - \mu(t)\frac{a(t)}{p(t)} - \left(\mu(0)y(0) - \mu(t)\frac{a(0)}{p(0)}\right) = -\int_0^t \frac{a(s)}{p(s)}\mu'(s)\,ds.$$

Because $y(0) = a(0)/p(0)$ and $\mu'(s) = -p(s)q(s)\mu(s)$, this simplifies to

$$y(t) = \frac{a(t)}{p(t)} + \frac{1}{\mu(t)}\int_0^t \mu(s)q(s)a(s)\,ds.$$

Thus the solution of (1.1) is

$$x(t) = a(t) + \frac{p(t)}{\mu(t)}\int_0^t \mu(s)q(s)a(s)\,ds. \tag{1.2}$$

Formula (1.2) is derived in Linz's monograph [13, p. 8]. However, to conform to a generalization of this result appearing later in this paper, we write (1.2) in another form. As $p(t)q(s) = B(t, s)$ and

$$\frac{\mu(s)}{\mu(t)} = e^{\int_s^t p(u)q(u)\,du} = e^{\int_s^t B(u,u)\,du},$$

we can express (1.2) in terms of the kernel B as follows:

$$x(t) = a(t) + \int_0^t B(t, s)\,e^{\int_s^t B(u,u)\,du}a(s)\,ds.$$

This suggests the first theorem. Its proof shows that our initial supposition about the differentiability of a, p, and q can be weakened to continuity.

Theorem 1.1. *Let a, p, and q be continuous on an interval $[0, T]$. For the separable kernel $B(t, s) = p(t)q(s)$, define*

$$R(t, s) := B(t, s)e^{\int_s^t B(u,u)\,du}. \tag{1.3}$$

Then

$$x(t) := a(t) + \int_0^t R(t, s)a(s)\,ds \tag{1.4}$$

is the unique continuous solution of (1.1) on $[0, T]$.

Proof. Define the function y by

$$y(t) := a(t) + \int_0^t R(t, s)a(s)\,ds,$$