Applied Mathematics and Computation 219 (2013) 11278–11290

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Approximation schemes for fuzzy stochastic integral equations

Marek T. Malinowski

Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Szafrana 4a, 65-516 Zielona Góra, Poland

ARTICLE INFO

Keywords:
Uncertainty
Fuzzy stochastic integral equation
Stochastic fuzzy differential equation
Fuzzy stochastic trajectory integral
Approximate solution

ABSTRACT

We consider fuzzy stochastic integral equations with stochastic Lebesgue trajectory integrals and fuzzy stochastic Itô trajectory integrals. Some methods of construction of approximate solutions to such the equations are examined. We study the Picard type approximations, the Carathéodory type approximations and the Maruyama type approximations of solutions. In considered framework, the solutions and approximate solutions are mappings with values in the space of fuzzy sets with basis of square integrable random vectors. Under Lipschitz and linear growth conditions each sequence of considered approximate solutions converges to the exact unique solution of the given fuzzy stochastic integral equation. For each type of approximate solutions, we show that the sequence of approximations is uniformly bounded and obtain some bounds for a distance between nth approximation and exact solution.

1. Introduction

Thinking about a physical problem which is transformed into a deterministic problem of ordinary differential equations we cannot usually be sure that this modeling is perfect because a typical feature of real-world phenomena is uncertainty. This term is mostly understood as stochastic uncertainty and methods of probability theory and stochastic analysis are utilized in its investigations. A great number of real-world phenomena in control theory, physics, biology, economics can be modeled by stochastic dynamical systems whose evolution is governed by random forces. Stochastic differential equations [2,21,25] constitute a mathematical tool for dealing with such phenomena.

On the other hand, it is commonly accepted that uncertainty can come from other sources than randomness. For example, this takes place if we have incomplete or vague informations on parameters of a considered dynamic system. The data of modeled systems can be imperfect due to a lack of measurement precision. This type of uncertainty is not stochastic. It is connected with an imprecision of human knowledge rather than with occurrence of random events. This kind of uncertainty is called epistemic uncertainty (vagueness, imprecision, ambiguity, softness, fuzziness).

The coexistence of stochastic and epistemic uncertainty in dynamic systems motivate to look for some mathematical tools which could be appropriate in description of evolution of such the hybrid systems. The fuzzy stochastic differential equations can be adequate in modeling of the dynamics of phenomena which are subjected to two kinds of uncertainties: randomness and fuzziness, simultaneously (see [9–13,16–19,24]).

In the studies of fuzzy stochastic differential equations, a main problem is a concept of fuzzy stochastic Itô integral which should cover the notion of classical stochastic Itô integral. In [12,16,17] one can find of two different approaches in defining fuzzy stochastic integral which can successfully be applied in the studies of stochastic fuzzy differential equations. Basing on the notion of set-valued stochastic Itô trajectory integral (see [6,7,22,23]) used in the problems of stochastic inclusions, fuzzy stochastic Itô integral is treated in [16] as a fuzzy set of the space of square integrable random vectors. This approach is also

E-mail address: M.Malinowski@wmie.uz.zgora.pl

0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2013.05.040

exploited in [14,18,19] in the studies of existence of solutions to fuzzy stochastic integral equations. The solutions to such equations are not fuzzy stochastic processes but just some mappings taking values in the set of fuzzy sets of the space of square integrable random vectors. In this way randomness is incorporated in the values of solutions.

In [12,17] the diffusion part of stochastic fuzzy differential equations is an embedding of real Itô integral into fuzzy numbers space. The solutions obtained by usage of this approach are some fuzzy stochastic processes. In [13,15] one can also find the further studies in this line.

In this paper, working in the framework established in [16], we consider three types of approximate solutions to fuzzy stochastic integral equations. They are Picard, Carathéodory and Maruyama approximations. Under Lipschitz condition and linear growth condition imposed on the equation coefficients, we show that each sequence of approximate solutions converge to the exact solution of the equation. Some uniform bounds for each sequence of approximations are established. Also, we present some estimations of distance between nth approximate solution and exact solutions, respectively to each of the three types of considered approximations. Since every ordinary set is also a fuzzy set, all the results of this paper apply to set-valued stochastic integral equations studied as in [16].

2. Preliminaries

In this section we collect a background material needed in context of solutions to fuzzy stochastic integral equations.

Let $\mathcal{X}$ be a separable Banach space, $\mathcal{K}_c^b(\mathcal{X})$ the family of all nonempty closed bounded and convex subsets of $\mathcal{X}$. The Hausdorff metric $H_{\mathcal{X}}$ in $\mathcal{K}_c^b(\mathcal{X})$ is defined by

$$H_{\mathcal{X}}(A,B) = \max\left\{\sup_{a\in A}\ \mathrm{dist}_{\mathcal{X}}(a,B), \sup_{b\in B}\ \mathrm{dist}_{\mathcal{X}}(b,A)\right\},$$

where $\mathrm{dist}_{\mathcal{X}}(a,B) = \inf_{b\in B}\|a-b\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$ denotes a norm in $\mathcal{X}$.

It is known (see [3]) that $(\mathcal{K}_c^b(\mathcal{X}), H_{\mathcal{X}})$ is a complete metric space.

Let $(U,\mathcal{U},\mu)$ be a measure space. Recall that a set-valued mapping $F : U \to \mathcal{K}_c^b(\mathcal{X})$ is said to be measurable if it satisfies:

$$\{u\in U : F(u)\cap C \neq \emptyset\} \in \mathcal{U} \quad \text{for every closed set} \quad C\subset \mathcal{X}.$$

A measurable multifunction F is said to be L^p-integrably bounded ($p \geqslant 1$), if $u \mapsto H_{\mathcal{X}}(F(u),\{0\})$ belongs to $L^p(U,\mathcal{U},\mu;\mathbb{R})$.

Denote $I=[0,T]$, where $T<\infty$. Let $(\Omega,\mathcal{A},\{\mathcal{A}_t\}_{t\in I},P)$ be a complete filtered probability space satisfying usual hypotheses, i.e. $\{\mathcal{A}_t\}_{t\in I}$ is an increasing and right continuous family of sub-σ-algebras of $\mathcal{A}$ and $\mathcal{A}_0$ contains all P-null sets.

Let $\{B(t)\}_{t\in I}$ be an $\{\mathcal{A}_t\}$-Brownian motion. We put $U = I\times\Omega$, $\mathcal{U}=\mathcal{N}$, where $\mathcal{N}$ denotes the σ-algebra of the nonanticipating elements in $I\times\Omega$, i.e.

$$\mathcal{N} = \{A\in\beta_I\otimes\mathcal{A} : A^t\in\mathcal{A}_t \quad \text{for every} \quad t\in I\},$$

where β_I is the Borel σ-algebra of subsets of I and $A^t=\{\omega : (t,\omega)\in A\}$. Finally we set $\mu=\lambda\times P$ as a measure, where λ is the Lebesgue measure on (I,β_I).

A d-dimensional stochastic process $f : I\times\Omega\to\mathbb{R}^d$ is called nonanticipating if f is $\mathcal{N}$-measurable.

Consider the space

$$L^2_{\mathcal{N}}(\lambda\times P) := L^2(I\times\Omega,\mathcal{N},\lambda\times P;\mathbb{R}^d).$$

Then for every $f\in L^2_{\mathcal{N}}(\lambda\times P)$ and $\tau,\ t\in I,\ \tau<t$ the Itô stochastic integral $\int_\tau^t f(s)dB(s)$ exists (cf. [2,21,25]) and one has $\int_\tau^t f(s)dB(s)\in L^2(\Omega,\mathcal{A}_t,P;\mathbb{R}^d)\subset L^2(\Omega,\mathcal{A},P;\mathbb{R}^d)$.

Let $F : I\times\Omega\to\mathcal{K}_c^b(\mathbb{R}^d)$ be a set-valued stochastic process, i.e. a family $\{F(t)\}_{t\in I}$ of $\mathcal{A}$-measurable set-valued mappings $F(t):\Omega\to\mathcal{K}_c^b(\mathbb{R}^d), t\in I$. We call F nonanticipating if it is $\mathcal{N}$-measurable. Let us define the set

$$S^2_{\mathcal{N}}(F,\lambda\times P) := \{f\in L^2_{\mathcal{N}}(\lambda\times P) : f\in F,\ \ \lambda\times P\text{-a.e.}\}.$$

If F is $L^2_{\mathcal{N}}(\lambda\times P)$-integrably bounded, then by Kuratowski and Ryll-Nardzewski selection Theorem (see e.g. [5]) it follows that $S^2_{\mathcal{N}}(F,\lambda\times P)\neq\emptyset$. Hence for every $\tau,\ t\in I,\ \tau<t$ we can define the set-valued trajectory Itô stochastic integral

$$\int_\tau^t F(s)dB(s) := \left\{\int_\tau^t f(s)dB(s) : f\in S^2_{\mathcal{N}}(F,\lambda\times P)\right\}.$$

By this definition we have $\int_\tau^t F(s)dB(s)\subset L^2(\Omega,\mathcal{A}_t,P;\mathbb{R}^d)$.

In the rest of the paper, for the sake of convenience, we will write L^2 instead of $L^2(\Omega,\mathcal{A},P;\mathbb{R}^d)$ and L^2_t instead of $L^2(\Omega,\mathcal{A}_t,P;\mathbb{R}^d)$ where $t\in I$.

Now we consider the set-valued stochastic Aumann trajectory integral. Similarly as in the preceding considerations, let $F : I\times\Omega\to\mathcal{K}_c^b(\mathbb{R}^d)$ be a nonanticipating and $L^2_{\mathcal{N}}(\lambda\times P)$-integrably bounded set-valued stochastic process. Then for $\tau,\ t\in I,\ \tau<t$ we define set-valued stochastic Aumann trajectory integral $\int_\tau^t F(s)ds$ as a subset of $L^2(\Omega,\mathcal{A}_t,P;\mathbb{R}^d)$ and described by