Applied Mathematics and Computation 219 (2013) 10655–10667

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Asymptotically non-expansive self-maps and global stability with ultimate boundedness of dynamic systems

M. De la Sen [a,*], A. Ibeas [b]

[a] Institute of Research and Development of Processes, Campus of Leioa, Barrio Sarriena, 48090 Leioa, Bizkaia, Spain
[b] Department of Telecommunications and Systems Engineering, Universitat Autonoma de Barcelona, 08193 Bellaterra, Barcelona, Spain

ARTICLE INFO

Keywords:
Contractive maps
Non-expansive maps
Metric space
Fixed points

ABSTRACT

This paper investigates self-maps $T : X \to X$ which satisfy a distance constraint in a metric space with mixed point-dependent non-expansive properties or, in particular, contractive ones, and potentially expansive properties related to some distance threshold. The above mentioned constraint is feasible in certain real-world problems of usefulness, for instance, when discussing ultimate boundedness in dynamic systems which guarantees Lyapunov stability. This fact makes the proposed analysis to be potentially useful to investigate global stability properties in dynamic systems in the potential presence of some locally unstable equilibrium points. The results can be applied to stability problems of dynamics systems and circuit theory as the given examples suggest.

1. Introduction

Fixed point theory and related techniques are of increasing interest for solving a wide class of mathematical problems where convergence of a trajectory or sequence to some equilibrium set is essential. Recently, the subsequent set of more sophisticated related problems are under strong research activity:

(1) In the, so-called, p-cyclic non-expansive or contractive self-maps map each element of a subset A_i of an either metric or Banach space $\mathbf{B}$ to an element of the next subset A_{i+1} in a strictly ordered chain of p subsets of $\mathbf{B}$ such that $A_{p+1} = A_1$. If the subsets do not intersect then fixed points do not exist and their relevance in Analysis is played by best proximity points [1,2]. Best proximity points are also of interest in hyperconvex metric spaces [3,4].
(2) The so-called Kannan maps are also being intensively investigated in the last years as well as their relationships with contractive maps. See for instance [5,6,11].
(3) Although there is an increasing number of theorems about fixed points in Banach or metric spaces, new related recent results have been proven. Some of those novel results are, for instance, the generalization in [7] of Edelstein fixed point theorem for metric spaces by proving a new theorem. Also, an iterative algorithm for searching a fixed point in non-expansive mappings in Hilbert spaces has been proposed in [8]. On the other hand, an estimation of the size of an attraction ball to a fixed point has been provided in [9] for nonlinear differentiable maps.
(4) Fixed point theory can be also used successfully to find oscillations of solutions of differential or difference equations which can be themselves characterized as fixed points. See, for instance [9,10,12,13]. The fixed point tools are also useful to investigate certain applied problems as, for instance, some ones related to image restoration [14].

* Corresponding author.
E-mail addresses: manuel.delasen@ehu.es (M. De la Sen), Asier.ibeas@uab.cat (A. Ibeas).

0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2013.04.009

(5) Recent research has been performed in [15] concerning the existence of common fixed points in non-self and non-asymptotically non-expansive mappings. On the other hand, Hyers–Ulam-type stability has been proven in [16] for a general mixed additive-quadratic-cubic-quartic functional equation and fixed point theory has been applied to fuzzy-type characterizations in [17]. On the other hand, relevant recent investigation is being addressed towards the study of mixed equilibrium problems linked to the set of non-unique fixed points in nonexpansive mappings [18,19].
(6) The robust stability of uncertain dynamic systems, potentially involving switching in-between different parameterizations, is a very important property to be achieved in most of applications which is often addressed through Lyapunov stability theory often with parallel use of related matrix inequalities. See, for instance, [13,20–24] and references there in. Stability is a basic property or requirement in many real life problems. See, for instance [23–31] concerning the stability of time-delayed, discrete, fractional and hybrid continuous-time/discrete-time systems and adaptive control as well as references there in. It is also a basic property to be guaranteed by the controller designer in common adaptive control problems and also in the analysis of the properties of equilibrium points and study of boundedness of the solutions in epidemic models and a number of real uncertain dynamic systems (see, for instance, [32–37] and references therein). In this context, one of the weakest, but at the same time useful, global stability concepts for dynamic systems with a unique equilibrium point is that of global Lyapunov stability with ultimate boundedness. Intuition dictates that the map defining the solution trajectory from any set of bounded initial conditions might be in some cases expansive close to the equilibrium point and contractive far away from the equilibrium point, thus preserving global stability (in the sense of boundedness of any solution from any bounded set of initial conditions). The dynamic system exhibits also local instability around the equilibrium point which is not an attractor since the mapping defining the trajectory is not contractive and it is not even non-expansive close to the equilibrium point.
(7) There are also interesting results available concerning the use of Fixed Point Theory in variational analytic and numerical methods and, in particular, in iterative methods based on variational inequalities. Variational inequalities have well-known relevant application fields in Physics and Control Theory based on Hamiltonian formalisms as it is the case, for instance, in what is concerning with variational min/max principles in Optics and Mechanics and in control optimization problems. It is obvious that the Physical variational principles are related to stability since the optimal trajectories are such that they make extremal a Hamiltonian-type functional involved. In control theory, the optimal control stabilizes the closed-loop system in parallel with making extremal the particular Hamiltonian defining the problem. A set of results concerning those topics can be found in [38–45] and references therein. Thus, there is some close research objectives between stability/stabilization studies and associated methods and variational-type principles in many disciplines.

Therefore, it turns out that there are many real-life problems which are characterized by self-maps which can be expansive, non-expansive or contractive depending of the size of the distance values within given sets as, for instance, the generation of state trajectory solutions from initial conditions in dynamic systems subject to disturbances of local large sizes (in terms of norms or distances) while such sizes are upper-bounded by asymptotically non-strictly increasing functions of the state trajectory solution norms. In this case, the system can exhibit simultaneous local instability around equilibrium points with global norms. In this case the system can exhibit simultaneous local instability around equilibrium points with global stability under the ultimate boundedness property. In other words, and roughly speaking, a tendency of the state-norm to diverge is neutralized by a contractive property of the state-trajectory solution for associated large state norms or, in Lyapunov stability theory terms, by the property of the Lyapunov functional candidate to become strictly decreasing for such large norms. This manuscript is devoted to investigate self-maps $T : X \to X$ in a metric space (X, d), or in a normed space $(X, \| \ \|)$, which satisfy the constraint $d(Tx, Ty) - d(x, y) \leqslant -Kd(x, y) + M$, for some real constants $K \geqslant 0$, $M \geqslant 0$. A motivating practical usefulness of this property is concerned, for instance, with the description of ultimate boundedness in stability dynamic systems. In particular, assume that $x, y = Tx$, $z = Ty = T^2x$ are three selected distinct points of a state-space trajectory solution of the dynamic system, ordered according to increasing time, such that the distance in-between the first and second points $d(x, y)$ exceeds a sufficiently large threshold. Then, under the above constraint, there is an ultimate boundedness property guaranteeing global stability, since the distance in-between the second and third points $d(y, z)$ decreases avoiding divergence with time of the state-trajectory solution. More formally, it is direct to see that $d(Tx, Ty) \leqslant d(x, y)$; i.e., $T : X \to X$ is non-expansive, if $d(x, y) \geqslant M/K$; $\forall x, y \in X$. Also, if X is bounded then

$$d(x,y) < M/K \Rightarrow d(Tx,Ty) \leqslant (1-K)d(x,y) + M < M/K; \quad \forall x,y \in X \tag{1.1}$$

Then, the self-map $T : X \to X$ exhibits the following constraint under (1.1) provided that it is continuous: $T : A_{x\,y} \to A_{\omega z}$ where $A_{x\,y} \subset X$ is the open circle of center $c_{x\,y} \in X$of radius $R := M/K$ for each given $x, y \in A_{x\,y}$and $A_{\omega z} \subset X$is an open circle of center at some $c_{\omega z} \in X$ also of radius R. Note that $A_{x\,y}$ can be distinct from $A_{\omega z}$. However, if $T : X \to X$ is not continuous then the existence of the above circles is not ensured; it is only known that (1.1) holds. Note that (1.1) does not guarantee that the self-map $T : X \to X$ is globally non-expansive since it can be expansive for small distances fulfilling $d(x, y) < M/K$ and non-expansive if $d(x, y) \geqslant M/K$. This elementary idea is addressed in the following simple technical proposition: