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On two classes of mixed-type Lyapunov equations ☆

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ABSTRACT

The Hermitian positive definite solutions of the mixed-type Lyapunov equations $X = AXB^* + BXA^* + Q$ and $AX + XA^* + BXB^* + Q = 0$ are studied in this paper. Based on the Bhaskar and Lakshmikantham's fixed point theorem, new sufficient conditions for the existence of Hermitian positive definite solutions are derived. Iterative methods are proposed to compute the Hermitian positive definite solutions. Numerical examples are used to illustrate the convergence of the new methods.

1. Introduction

In this paper, we consider the following two classes of mixed-type Lyapunov equations

$$X = AXB^* + BXA^* + Q, \tag{1.1}$$

and

$$AX + XA^* + BXB^* + Q = 0, \tag{1.2}$$

where A, B are $n \times n$ complex matrices, Q is an $n \times n$ Hermitian positive definite matrix and the Hermitian positive definite solution X is of practical interest. The mixed-type Lyapunov equation (1.1) arises in solving some nonlinear matrix equations (see Xu and Chen [23] and Sakhnovich [20] for more details). For example, if the Newton's method is used to solve the nonlinear matrix equation $Y - C^*Y^{-2}C = I$, the following equation

$$Y_{k+1} + C^*Y_k^{-2}Y_{k+1}Y_k^{-1}C + C^*Y_k^{-1}Y_{k+1}Y_k^{-2}C = I + 3C^*Y_k^{-2}C, \tag{1.3}$$

must be solved in each iterative step. Set

$$A = C^*Y_k^{-2}, \quad B = -C^*Y_k^{-1}, \quad X = Y_{k+1} - I,$$

then Eq. (1.3) can be rewritten as Eq. (1.1), where $Q = C^*\left(3Y_k^{-2} - 2Y_k^{-3}\right)C$. Eq. (1.2) arises in the model order reduction of bilinear control systems and the stochastic H_2/H_∞ control with state-dependent noise (see [4,6,9,26]). The positive definite solution of Eq. (1.2) can be interpreted as the controllability Gramian of the bilinear control system

$$\dot{x}(t) = Ax(t) + Bx(t)u(t) + Wu(t).$$

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For the mixed-type Lyapunov equation (1.1), Xu and Chen [23] gave some necessary and sufficient conditions for the existence of a positive definite solution by using Lyapunov operator. Recently, Cheng and Xu [5] studied the perturbation analysis of Eq. (1.1), and derived some sharp perturbation bounds for the positive definite solution. Two special cases of Eq. (1.1) have been extensively investigated in the past 30 years. If $B = \frac{1}{2}A$, the mixed-type Lyapunov equation (1.1) is the discrete Lyapunov equation

$$X = AXA^* + Q, \tag{1.4}$$

and if $B = I, \widetilde{A} = A - \frac{1}{2}I$, then Eq. (1.1) is the continuous Lyapunov equation

$$\widetilde{A}X + X\widetilde{A}^* + Q = 0. \tag{1.5}$$

Many numerical methods, such as HSS iterative method [3], gradient method [7,8,22,24,25], Bartels–Stewart method [12,13], Hessenberg–Schur method [11], multigrid algorithm [10], block GMRES–Sylvester method [17] and kronecker product iterative method [21], were proposed to solve Eqs. (1.4) and (1.5).

For the mixed-type Lyapunov equation (1.2), Benner and Damm [4] gave some sufficient conditions for the existence of a positive definite solution by the linear operator theory and its spectral theory. However, it is difficult to verify these sufficient conditions (see Section 3 for more details). By making use of the vectorizing operator and Kronecker product, Huang [15,16] transformed Eq. (1.2) into a system of linear equations and derived some sufficient conditions for the existence of a symmetric solution. A parameter iterative method was constructed to compute the symmetric solution, but it is unknown to choose the optimal parameter.

Motivated by the works and applications in [5,6,9,23,26], we continue to study the positive definite solution of the mixed-type Lyapunov equations (1.1) and (1.2) in this paper. By making use of the Bhaskar and Lakshmikantham's fixed point theorem, we give some new sufficient conditions for the existence of positive definite solutions of Eqs. (1.1) and (1.2), which are easier to verify than those in [4,23]. Iterative methods are proposed to solve them, and the error estimations are also given. Some numerical examples are presented to show that the iterative methods are feasible.

Throughout this paper, we write $B > 0$ $(B \geqslant 0)$ if the matrix B is positive definite (semidefinite). If $B - C$ is positive definite (semidefinite), then we write $B > C$ $(B \geqslant C)$. If a positive definite matrix X satisfies $B \leqslant X \leqslant C$, we denote by $X \in [B, C]$. We use $\lambda_1(B)$ and $\lambda_n(B)$ to denote the maximal and minimal eigenvalues of the Hermitian matrix B, respectively. We use $C^{n\times n}, H(n)$ and $P(n)$ to stand for the set of all $n \times n$ complex matrices, all $n \times n$ Hermitian matrices and all $n \times n$ Hermitian positive definite matrices, respectively. The symbol I stands for the $n \times n$ identity matrix. The symbols $tr(B)$ and B^* denote the trace and the conjugate transpose of the matrix B, respectively. We denote the spectral norm by $\|\cdot\|$, i.e. $\|A\| = \sqrt{\lambda_1(A^*A)}$. For $Q \in P(n)$, the Q-norm is defined by

$$\|A\|_Q = tr(Q^{\frac{1}{2}}AQ^{\frac{1}{2}}).$$

From Theorem IX.2.2 in [14] we know that $H(n)$ equipped with the metric induced by $\|\cdot\|_Q$ is a complete metric space for any $Q \in P(n)$.

2. Hermitian positive definite solutions of Eqs. (1.1) and (1.2)

In this section, we derive some new sufficient conditions for the mixed-type Lyapunov equations (1.1) and (1.2). Two iterative methods are proposed to solve them, and the error estimations are also given.

Lemma 2.1 ([18]). *Let $A \geqslant 0$ and $B \geqslant 0$ be $n \times n$ matrices, then $0 \leqslant tr(AB) \leqslant \|A\| tr(B)$.*

Lemma 2.2 (Bhaskar and Lakshmikantham's fixed point theorem, [2]). *Let $(W, \leqslant)$ be the partially ordered set and d be a metric on W such that (W, d) is a complete metric space. Let the map $F : W \times W \to W$ be continuous and mixed monotone on W. Assume that there exists a $\delta \in [0, 1)$ with*

$$d(F(x,y), F(u,v)) \leqslant \frac{\delta}{2}[d(x,u) + d(y,v)],$$

for all $x \geqslant u$ and $y \leqslant v$. Suppose also that

(i) there exist $x_0, y_0 \in W$ such that $x_0 \leqslant F(x_0, y_0)$ and $y_0 \geqslant F(y_0, x_0)$;
(ii) every pair of elements has either a lower bound or an upper bound, that is, for every $(x,y) \in W \times W$, there exist z_1 and z_2 such that $x, y \leqslant z_1$ or $x, y \geqslant z_2$. Then there exists a unique $\bar{x} \in W$ such that $\bar{x} = F(\bar{x}, \bar{x})$. Moreover, the sequences $\{x_k\}$ and $\{y_k\}$ generated by $x_{k+1} = F(x_k, y_k)$ and $y_{k+1} = F(y_k, x_k)$ converge to $\bar{x}$, with the following estimate

$$\max\{d(x_k, \bar{x}), d(y_k, \bar{x})\} \leqslant \frac{\delta^k}{1-\delta}\max\{d(x_1, x_0), d(y_1, y_0)\}.$$