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Dynamics of a family of Chebyshev–Halley type methods ☆

Alicia Cordero [a], Juan R. Torregrosa [a,*], Pura Vindel [b]

[a] Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera s/n, 46022 València, Spain
[b] Instituto de Matemáticas y Aplicaciones de Castellón, Universitat Jaume I, Campus de Riu Sec s/n, Castellón, Spain

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ABSTRACT

In this paper, the dynamics of the Chebyshev–Halley family is studied on quadratic polynomials. A singular set, that we call cat set, appears in the parameter space associated to the family. This set has interesting similarities with the Mandelbrot set. The parameter space has allowed us to find different elements of the family which have bad convergence properties, since periodic orbits and attractive strange fixed points appear in the dynamical plane of the corresponding method.

1. Introduction

The application of iterative methods for solving nonlinear equations $f(z) = 0$, with $f : \mathbb{C} \to \mathbb{C}$, gives rise to rational functions whose dynamics are not well-known. The simplest model is obtained when $f(z)$ is a quadratic polynomial and the iterative process is Newton's method. The study of the dynamics of Newton's method has been extended to other point-to-point iterative methods used for solving nonlinear equations, with convergence order up to three (see, for example [1,2] and, more recently, [3,4]).

The most of the well-known point-to-point cubically convergent methods belong to the one-parameter family, called Chebyshev–Halley family,

$$z_{n+1} = z_n - \left(1 + \frac{1}{2}\frac{L_f(z_n)}{1 - \alpha L_f(z_n)}\right)\frac{f(z_n)}{f'(z_n)}, \tag{1}$$

where

$$L_f(z) = \frac{f(z)f''(z)}{(f'(z))^2} \tag{2}$$

and α is a complex parameter. This family includes Chebyshev's method for $\alpha = 0$, Halley's scheme for $\alpha = \frac{1}{2}$, super-Halley's method for $\alpha = 1$ and Newton's method when α tends to $\pm\infty$. As far as we know, this family was already studied by Werner in 1981 (see [5]), and can also be found in [6,7]. Moreover, a geometrical construction of this family is studied in [8]. It is interesting to note that any iterative process given by the expression:

$$z_{n+1} = z_n - H(L_f(z_n)), \tag{3}$$

where function H satisfies $H(0) = 0,\ H'(0) = \frac{1}{2}$ and $|H''(0)| < \infty$, generates iterative methods of order three (see [9]).

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* Corresponding author.
E-mail addresses: acordero@mat.upv.es (A. Cordero), jrtorre@mat.upv.es (J.R. Torregrosa), vindel@uji.es (P. Vindel).

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The family of Chebyshev–Halley has been widely analyzed under different points of view. For example, in [10–12], the authors studied the conditions under which the global and semilocal convergence of this family in Banach spaces hold. The semilocal convergence of this family in the complex plane is also presented in [13].

Many authors have introduced different variants of this family, in order to increase its applicability and its order of convergence. For instance, Osada in [14] showed a variant able to find the multiple roots of analytic functions and a procedure to obtain simultaneously all the roots of a polynomial. On the other hand, in [15–17] the authors got multipoint variants of the mentioned family with sixth order of convergence. Another trend of research about this family have been to avoid the use of second derivatives (see [18–20]) or to design secant-type variants (see [21]).

From the numerical point of view, the dynamical behavior of the rational function associated with an iterative method give us important information about its stability and reliability. In these terms, Varona in [22] described the dynamical behavior of several well-known iterative methods. More recently, in [3,23–30], the authors study the dynamics of different iterative families.

The fixed point operator corresponding to the family of Chebyshev–Halley described in (1) is:

$$G(z) = z - \left(1 + \frac{1}{2}\frac{L_f(z)}{1-\alpha L_f(z)}\right)\frac{f(z)}{f'(z)}. \tag{4}$$

In this work, we study the dynamics of this operator when it is applied to quadratic polynomials. It is known that the roots of a polynomial can be transformed by an affine map with no qualitative changes on the dynamics of family (1) (see [31]). So, we can use the quadratic polynomial $p(z) = z^2 + c$. For $p(z)$, the operator (4) corresponds to the rational function:

$$G_p(z) = \frac{z^4(-3+2\alpha) + 6cz^2 + c^2(1-2\alpha)}{4z(z^2(-2+\alpha)+\alpha c)}, \tag{5}$$

depending on the parameters $\alpha$and c.

Blanchard, in [32], by considering the conjugacy map

$$h(z) = \frac{z - i\sqrt{c}}{z + i\sqrt{c}}, \tag{6}$$

with the following properties:

$$\text{(i) } h(\infty) = 1, \quad \text{(ii) } h(i\sqrt{c}) = 0, \quad \text{(iii) } h(-i\sqrt{c}) = \infty,$$

proved that, for quadratic polynomials, the Newton's operator is always conjugate to the rational map z^2. In an analogous way, it is easy to prove, by using the same conjugacy map, that the operator $G_p(z)$ is conjugated to the operator $O_p(z)$

$$O_p(z) = \left(h \circ G_p \circ h^{-1}\right)(z) = z^3\frac{z - 2(\alpha-1)}{1 - 2(\alpha-1)z}. \tag{7}$$

We observe that the parameter c has been obviated in $O_p(z)$.

In this work, we study the general convergence of methods (1) for quadratic polynomials. To be more precise (see [33,34]), a given method is generally convergent if the scheme converges to a root for almost every starting point and for almost every polynomial of a given degree.

Now, we are going to recall some dynamical concepts of complex dynamics (see [35]) that we use in this work. Given a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$\{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\}.$$

We are interested in the study of the asymptotic behavior of the orbits depending on the initial condition z_0, that is, we are going to analyze the phase plane of the map R defined by the different iterative methods. To obtain these phase spaces, we classify the starting points from the asymptotic behavior of their orbits.

A $z_0 \in \hat{\mathbb{C}}$ is called a *fixed point* if $R(z_0) = z_0$. A *periodic point* z_0 of period $p > 1$ is a point such that $R^p(z_0) = z_0$ and $R^k(z_0) \neq z_0$, for $k < p$. A *pre-periodic point* is a point z_0 that is not periodic but there exists a $k > 0$ such that $R^k(z_0)$ is periodic. A *critical point* z_0 is a point where the derivative of the rational function vanishes, $R'(z_0) = 0$. A fixed point z_0 is called *attractor* if $|R'(z_0)| < 1$, *superattractor* if $|R'(z_0)| = 0$, *repulsor* if $|R'(z_0)| > 1$ and *parabolic* if $|R'(z_0)| = 1$.

The basin of attraction of an attractor α is defined as the set of pre-images of any order:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to \alpha,\ n \to \infty\}\leftarrow$$

The Fatou set of the rational function $R, \mathcal{F}(R)$, is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\hat{\mathbb{C}}$ is the *Julia set*, $\mathcal{J}(R)$; therefore, the Julia set includes all repelling fixed points, periodic orbits and their pre-images. That means that the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

The invariant Julia set for Newton's method on quadratic polynomials is the unit circle S^1 and the Fatou set is defined by the two basins of attraction of the superattractor fixed points: 0 and ∞. Nevertheless, the Julia set for Chebyshev's method applied to quadratic polynomials is more complicated than for Newton's method and it has been studied in [36]. These meth-