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An upwind-like discontinuous Galerkin method for hyperbolic systems ☆

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ABSTRACT

We investigate an upwind-like DG method for solving first-order hyperbolic problems written as the Friedrichs' systems. Under certain condition, this DG scheme may be semi-explicit such that the discrete equations can be solved layer by layer. We give the stability analysis and error estimate of order $k+1/2$ in the DG-norm. In particular, for some hyperbolic systems, we show that the convergence rate is of order $k+1$ in the L_2-norm if the Q_k-elements are used on rectangular meshes. Finally, we provide some numerical experiments to illustrate the theoretical analysis.

1. Introduction

The discontinuous Galerkin (DG) finite element methods have attracted more and more attention in the field of numerical partial differential equations during the last decades, see [1,2] and the references therein. The main advantages of the DG method are the high order accuracy, the high degree of parallelism, and its great suitability for h, p, and hp refinements involved in adaptive computations. Historically, the original DG method was introduced by Reed and Hill [15] in 1973 to simulate the neutron transport equation, and the first theoretical analysis of DG methods for hyperbolic equation was performed by Lesaint and Raviart [12] in 1974. This analysis was subsequently improved by Johnson and Pitkaranta [9] who established that the optimal order of convergence in the L_2-norm is $k+1/2$ if piecewise polynomials of degree k are used. Peterson in [14] further proved that the convergence rate of order $k+1/2$ is sharp for DG methods within quasi-uniform triangulation. However, a better error estimate of order $k+1$ can also be achieved in at least two circumstances: the case of rectangular meshes [12] and the case of some structured triangular meshes, see [3,16].

Many DG methods have also been presented for solving the first-order hyperbolic problems written as the Friedrichs' systems,

$$\sum_{i=1}^{d} A_i \partial_i \mathbf{u} + B\mathbf{u} = \mathbf{f}, \quad \text{in } \Omega \subset R^d. \tag{1.1}$$

Basically these DG methods can be classified as both the numerical flux method and the penalty method, see [5,6,8,11,13,17,18]. In the numerical flux method, the key technique is to chose the numerical trace $D_n\hat{\mathbf{u}}$ properly in the weak form of problem (1.1)

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$$-\int_K \mathbf{u}\cdot\sum_{i=1}^{d} A_i\partial_i\mathbf{v} + \int_K \left(B-\sum_{i=1}^{d}\partial_i A_i\right)\mathbf{u}\cdot\mathbf{v} + \int_{\partial K} D_n\widehat{\mathbf{u}}\cdot\mathbf{v} = \int_K \mathbf{f}\cdot\mathbf{v}, \tag{1.2}$$

where matrix $D_n = \sum_{i=1}^{d} A_i n_i$, $n = (n_1,\ldots,n_d)^T$ is the outward unit normal vector on the element boundary ∂K. In the traditional upwind-like scheme (see [13,17,10,1]), the numerical trace is defined by first splitting matrix D_n into the symmetric form $D_n = A^+ + A^-$ with $A^+ \geqslant 0$ (positive semi-definite) and $A^- \leqslant 0$ (negative semi-definite), and then setting the numerical trace $D_n\widehat{\mathbf{u}} = A^+\mathbf{u}^+ + A^-\mathbf{u}^-$, where $\mathbf{u}^+$ and $\mathbf{u}^-$ are the traces of $\mathbf{u}$ on ∂K from the interior and exterior of K, respectively. In this paper, we will present an upwind-like DG scheme which is slightly different from the traditional one. We first decompose each A_i into $A_i = A_i^+ + A_i^-$, and then define the numerical trace by setting $D_n\widehat{\mathbf{u}} = \sum_{i=1}^{d} A_i^+ n_i\widehat{\mathbf{u}} + \sum_{i=1}^{d} A_i^- n_i\widehat{\mathbf{u}}$, and $A_i^\pm n_i\widehat{\mathbf{u}} = A_i^\pm n_i\mathbf{u}^+ (A_i^\pm n_i\mathbf{u}^-)$ if $A_i^\pm n_i \geqslant 0 (A_i^\pm n_i \leqslant 0)$. The advantages of our scheme are as follows. Firstly, the matrices only need to be split once before the triangulation is made, while in the traditional method, since matrix D_n depends on the boundary normal vector n, then for each element K and each face $\mathcal{F}_K \subset \partial K$, we always need to split $D_n|_{\mathcal{F}_K}$. Therefor, such splitting is very consuming in practical computations. Secondly, if $A_i \geqslant 0$ for some fixed i, our scheme will be explicit in the x_i-axis direction so that the discrete problem may be solved layer by layer along x_i-direction (see Section 2). For arbitrary shape-regular triangulations, we give the stability analysis and error estimate of order $k+1/2$ in the DG-norm which is stronger than the L_2-norm. In particular, under the assumption of all $A_i \geqslant 0$, we show that the convergence rate is of order $k+1$ in the L_2-norm if the Q_k-elements are used on rectangular meshes and the solution $\mathbf{u}$ is in $H^{k+2}(\Omega)$. To the authors' knowledge, the best error estimate of DG methods for hyperbolic systems now is of order $k+1/2$, so our here result is new and has some theoretical significance into the literature.

Throughout this paper, let Ω be a bounded open polyhedral domain in R^d, $d \geqslant 2$. For any open subset $\mathcal{D} \subset \Omega$ and integers $m \geqslant 0$, we denote by $H^m(\mathcal{D})$ the usual Sobolev spaces equipped with norm $\|\cdot\|_{m,\mathcal{D}}$ and semi-norm $|\cdot|_{m,\mathcal{D}}$, and denote by $(\cdot,\cdot)_{\mathcal{D}}$ and $\|\cdot\|_{0,\mathcal{D}}$ the standard inner product and norm in the space $H^0(\mathcal{D}) = L_2(\mathcal{D})$. When $\mathcal{D} = \Omega$, we omit the index $\mathcal{D}$. We will use letter C to represent a generic positive constant, independent of the mesh size h.

The plan of this paper is as follows. In Section 2, the DG method is analyzed and the stability is discussed. Section 3 is devoted to the error analysis in the DG-norm. In Section 4, we derive the optimal error estimate of order $k+1$ in the L_2-norm on rectangular meshes. Finally, in Section 5, we provide some numerical experiments to illustrate our theoretical analysis.

2. Problem and its DG approximation

Consider the following first-order hyperbolic system:

$$\mathcal{L}\mathbf{u} \equiv \boldsymbol{A}\cdot\nabla\mathbf{u} + B\mathbf{u} = \mathbf{f}, \quad x\in\Omega, \tag{2.1}$$

$$(M - D_n)\mathbf{u} = \mathbf{0}, \quad x\in\partial\Omega. \tag{2.2}$$

Here, $\boldsymbol{A} = (A_1,\ldots,A_d)^T$ is a vector matrix function, $\boldsymbol{A}\cdot\nabla\mathbf{u} = \sum_{i=1}^{d} A_i\partial_i\mathbf{u}$, A_i, B and M are some given $m\times m$ matrices, $A_i \in [W_\infty^1(\Omega)]^{m\times m}$, $B, M \in [L_\infty(\Omega)]^{m\times m}$, $D_n = \boldsymbol{A}\cdot n = \sum_{i=1}^{d} A_i n_i$, $n(x) = (n_1,\ldots,n_d)^T$ is the outward unit normal vector at the point $x\in\partial\Omega$, $\mathbf{u} = (u_1,\ldots,u_m)^T$ and $\mathbf{f} = (f_1,\ldots,f_m)^T$ with $f_i \in L_2(\Omega)$ are m-dimensional vector functions. We assume that problem (2.1)–(2.2) is a positive and symmetric hyperbolic system (Friedrichs' system [7]), namely,

$$A_i = A_i^T, \quad i = 1,\ldots,d, \ x\in\Omega, \tag{2.3}$$

$$B + B^T - div\boldsymbol{A} \geqslant 2\sigma_0 I, \quad x\in\Omega, \tag{2.4}$$

$$M + M^T \geqslant 0, \quad x\in\partial\Omega, \tag{2.5}$$

$$Ker(M - D_n) + Ker(M + D_n) = R^m, \quad x\in\partial\Omega, \tag{2.6}$$

where constant $\sigma_0 > 0$, $div\mathbf{A} = \partial_1 A_1 + \cdots + \partial_d A_d$, and by using the expression $A \geqslant 0 (\leqslant 0)$ we imply that the matrix A is positive (negative) semi-definite.

Problem (2.1)–(2.2) can describe many important physics processes. An example of such Friedrichs' system is as follows.

Maxwell's equations. Let σ and μ be two positive functions in $L_\infty(\Omega)$ uniformly bounded away from zero. Consider the following Maxwell's equations in R^3

$$\mu H + \nabla\times E = h, \quad x\in\Omega,$$

$$\sigma E - \nabla\times H = g, \quad x\in\Omega,$$

$$E\times n = 0, \quad x\in\partial\Omega,$$

where H and E are three-dimensional vector functions. This problem can be cast into the form of Friedrichs' system by setting $\mathbf{u} = (H, E)^T$,

$$A_i = \begin{pmatrix} O & Q_i \\ Q_i^T & O \end{pmatrix}_{,i=1,2,3,} \quad B = \begin{pmatrix} \mu I & O \\ O & \sigma I \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} h \\ g \end{pmatrix},$$