Applied Mathematics and Computation 219 (2013) 8779–8789

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Banach algebras of matrix transformations between some sequence spaces related to Λ-strong convergence and boundedness ☆

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ARTICLE INFO

Keywords:
Λ-Strong convergence and boundedness
Matrix transformations
Banach algebras
Hausdorff measure of noncompactness
Compact operators
Fredholm operators

ABSTRACT

We study the spaces $c_0(\Lambda), c(\Lambda)$ and $c_\infty(\Lambda)$ of sequences that are Λ-strongly convergent to zero, Λ-strongly convergent and Λ-strongly bounded, and the related spaces $v_0(\Lambda)$ and $v_\infty(\Lambda)$. In particular, we give the characterizations of several classes of matrix transformations between those spaces. Furthermore, we use our results to prove that the classes of matrix transformations from $v_\infty(\Lambda), c_\infty(\Lambda)$ and $c(\Lambda)$ into themselves are Banach algebras. As an application of our results, we establish an estimate for the Hausdorff measure of noncompactness of matrix operators that map $c(\Lambda)$ into itself, give a characterization of the subclass of compact operators, and a sufficient condition for those operators to be Fredholm.

1. Introduction

Mòricz [22] introduced the set $c(\Lambda)$ of Λ-strongly convergent sequences. This set and the sets $c_0(\Lambda)$ and $c_\infty(\Lambda)$ of sequences that are Λ-strongly convergent to zero and Λ-strongly bounded were studied in [15]. Here we also consider the sets $v_0(\Lambda)$ and $v_\infty(\Lambda)$ which are closely related to $c_0(\Lambda)$ and $c_\infty(\Lambda)$. The sets $v_0(\Lambda)$, $v_\infty(\Lambda), c_0(\Lambda)$ and $c_\infty(\Lambda)$ are the matrix domains of certain triangles in the spaces $w_0(\Lambda)$ and $w_\infty(\Lambda)$ that were recently studied in [19].

Here we characterize several classes of matrix transformations between the spaces $v_0(\Lambda)$, $v_\infty(\Lambda), c_0(\Lambda)$ and $c_\infty(\Lambda)$ by giving necessary and sufficient conditions on the entries of the infinite matrices. Furthermore, we apply our results to establish that the classes of matrix transformations from $v_\infty(\Lambda), c_\infty(\Lambda)$ and $c(\Lambda)$ into themselves are Banach algebras, thus generalizing the corresponding results in [16]. Finally, we apply our results to establish an estimate for the Hausdorff measure of noncompactness of matrix operators that map $c(\Lambda)$ into itself, and a sufficient condition for those operators to be Fredholm.

Characterizations of classes (X, Y) of matrix transformations from a sequence space X into a sequence space Y constitute a wide, interesting and important field in both summability and operator theory. These results, in particular, the so-called row norm conditions in the characterizations of the classes of matrix transformations, are needed, among other things, to establish identities or estimates for the Hausdorff measure of noncompactness of the corresponding operators. Also, these results are needed for the characterizations of the corresponding subclasses of compact matrix operators as, for instance, in

☆ Research of the authors supported by the research projects #174025 and #174007, respectively, of the Serbian Ministry of Science, Technology and Environmental Development

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http://dx.doi.org/10.1016/j.amc.2013.02.067

[1,28,18,25–27], and more recently, of general bounded linear operators between the respective sequence spaces as, for instance, in [4,6,12,7,2]. They are also applied in studies of the invertibility of operators and the solvability of infinite systems of linear equations and of linear differential equations as, for instance, in [11,13,12 and 3,23,24], respectively.

To be able to apply methods from the theory of Banach algebras to the solution of those problems it is essential to determine if a class of linear matrix operators of a sequence space X into itself is a Banach algebra. Although it is well-known [29] that if X is Banach space over $\mathbb{C}$, then the Banach space of bounded linear operators on X is Banach algebra with identity I, the corresponding result for matrix operators is non trivial if X is a BK space which does not have AK, as in our cases. It is also worth mentioning that the characterizations of compact operators can be used to establish sufficient conditions for an operator to be a Fredholm operator as, for instance, in [5]. We give an application of this kind in Section 5 for Fredholm operators from $c(\Lambda)$ into itself.

Here we give the characterizations the classes

$$(v_\infty(\Lambda), v_\infty(\Lambda')), (v_0(\Lambda), v_\infty(\Lambda'), (c_\infty(\Lambda), c_\infty(\Lambda')), (c_0(\Lambda), c_\infty(\Lambda')) \text{ and } (c(\Lambda), c(\Lambda)).$$

We also show that the classes $(v_\infty(\Lambda), v_\infty(\Lambda)), (c_\infty(\Lambda), c_\infty(\Lambda))$ and $(c(\Lambda), c(\Lambda))$ are Banach algebras with respect to the corresponding operator norms.

Finally we apply our results to establish sufficient conditions for a matrix operator in the class $(c(\Lambda), c(\Lambda))$ to be Fredholm.

2. Notations and known results

Here we list the standard notations and results that are needed in this paper.

2.1. Standard notations and basic results

Let ω denote the set of all complex sequences $x = (x_k)_{k=0}^\infty$, and ℓ_∞, c, c_0 and ϕ be the sets of all bounded, convergent, null and finite sequences; also let cs and ℓ_1 denote the sets of all convergent and absolutely convergent series. We write e and $e^{(n)}$ $(n = 0, 1, \ldots)$ for the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. Let $x, y \in \omega$. Then we write $x \cdot y = (x_k y_k)_{k=0}^\infty$; if y has no zero terms, then $x/y = (x_k/y_k)_{k=0}^\infty$, in particular, $1/y = e/y$.

A sequence (b_n) in a linear metric space X is called a Schauder basis for X if, for every $x \in X$, there exists a unique sequence (λ_n) of scalars such that $x = \sum_n \lambda_n b_n$.

A BK space is a Banach sequence space with continuous coordinates $P_n : X \to \mathbb{C} (n = 0, 1, \ldots)$ where $P_n(x) = x_n$ for all $x = (x_k)_{k=0}^\infty \in X$. A BK space $X \supset \phi$ is said to have AK if $x = \sum_{k=0}^\infty x_k e^{(k)}$ for each sequence $x = (x_k)_{k=0}^\infty \in X$.

Let X be a subset of ω, and z be a sequence. Then we write $z^{-1} * X = \{x \in \omega : x \cdot z \in X\}$. The set $X^\beta = \bigcap_{x \in X}(x^{-1} * cs) = \{a \in \omega : a \cdot x \in cs \text{ for all } x \in X\}$ is called the β-dual of X. We note that we obviously have for any sequence u which has no zero terms

$$\left(u^{-1} * X\right)^\beta = (1/u)^{-1} * X^\beta. \tag{2.1}$$

Let $A = (a_{nk})_{n,k=0}^\infty$ be an infinite matrix of complex numbers, $x \in \omega$, and X and Y be subsets of ω. We write $A_n = (a_{nk})_{k=0}^\infty$ $(n = 0, 1, \ldots)$ and $A^k = (a_{nk})_{n=0}^\infty$ $(k = 0, 1, \ldots)$ for the sequences in the nth row and the kth column of A, and $A_n x = \sum_{k=0}^\infty a_{nk} x_k$ and $Ax = (A_n x)_{n=0}^\infty$ provided the series converge for all n. The set $X_A = \{x \in \omega : Ax \in X\}$ is called the matrix domain of A in X. We write (X, Y) for the set of all infinite matrices A that map X into Y, that is, for which $X_A \subset Y$; we write $(X) = (X, X)$, for short. We note that if u and v are sequences which have no zero terms then we obviously have

$$A \in (u^{-1} * X, v^{-1} * Y) \text{ if and only if } B = (b_{nk})_{n,k=0}^\infty \in (X, Y) \text{ where } b_{nk} = \frac{v_n a_{nk}}{u_k} \text{ for } n, k = 0, 1, \ldots. \tag{2.2}$$

Let X and Y be Banach spaces and $B_X = \{x \in X : \|x\| \leqslant 1\}$ be the closed unit ball in X. We write $\mathcal{B}(X, Y)$ for the Banach space of all bounded linear operators $L : X \to Y$ with the operator norm $\|L\| = \sup\{\|L(x)\| : x \in B_X\}$ for all $L \in \mathcal{B}(X, Y)$; we write $\mathcal{B}(X) = \mathcal{B}(X, X)$, for short. As usual, $X^* = \mathcal{B}(X)\mathbb{C}$ denotes the continuous dual of X with the norm $\|f\| = \sup\{|f(x)| : x \in B_X\}$ for all $f \in X^*$.

The following definitions and results are well known. Since they will frequently be applied, they are stated here for the reader's convenience.

Let a be a sequence and X be a normed sequence space. Then we write

$$\|a\|_X^* = \sup_{x \in B_X} \left| \sum_{k=0}^\infty a_k x_k \right|$$

provided the expression on the righthand side exists and is finite which is the case whenever X is a BK space and $a \in X^\beta$ by [31, Theorem 7.2.9].