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Local quasi-interpolants based on special multivariate quadratic spline space over a refined quadrangulation

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ABSTRACT

In this paper, we describe the construction of a suitable normalized B-spline representation for special multivariate quadratic spline space $\mathcal{S}_2^{1,0}(\Delta)$ over a refined quadrangulation. We then develop as an application, a general theory of quasi-interpolants based on this representation.

1. Introduction

The flexibility of geometric modelling of complex surfaces relies crucially on the use of an appropriate mathematical representation [3]. A surface s is usually represented as a linear combination of basis functions ϕ_i,

$$s = \sum_{i=1}^{N} c_i \phi_i.$$

The surface can be locally controlled and edited in a predictable way when the basis functions $\phi_i,\ i = 1, \ldots, N$, have a local support and form a convex partition of unity, i.e.,

$$\phi_i \geqslant 0 \quad \text{and} \quad 1 = \sum_{i=1}^{N} \phi_i.$$

Continuity conditions can be imposed to obtain smooth surfaces.

The application of spline in numerical computation requires efficient algorithms for constructing locally supported bases for the spline spaces. Dierckx [2] presented a geometric method to construct a normalized basis for the space of Powell–Sabin quadratic splines. Speleers [8] developed a suitable normalized B-spline representation for $\mathcal{C}^2$-continuous quintic Powell–Sabin splines. Recently, Speleers [9] constructs a suitable normalized B-spline representation for reduced cubic Clough-Tocher splines.

In this paper we consider special multivariate quadratic spline space $\mathcal{S}_2^{1,0}(\Delta)$ over a refined quadrangulation. This space has been recently studied in [10]. More precisely, its dimension and the explicit representations of the Hermite basis splines are obtained by using the smoothing cofactor-conformality method. In this paper, we will construct a compact normalized basis for this space. The basis functions have a local support, they are nonnegative, and they form a partition of unity. The construction is based on the determination of a set of triangles that must contain a specific set of points. We are able to define control points and give Marsden identity for representing quadratic polynomials. Using some basic properties of the

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blossoming principle, we also show how to construct discrete and differentiable quasi-interpolant which reproduce quadratic polynomials.

The paper is organized as follows. In Section 2 we review some general concepts of polynomials on triangles, and we present the multivariate spline space $\mathcal{S}_2^{1,0}(\Delta)$. Section 3 covers the construction of a normalized B-spline basis. In Section 4, we develop a general theory of quasi-interpolants based on this representation. Differential and discrete quasi-interpolants are constructed. Finally, in order to illustrate our results, we give in Section 5 some numerical examples.

2. Multivariate spline space $\mathcal{S}_2^{1,0}(\Delta)$

2.1. Blossoming

In this subsection, we review some basic properties of the blossoming principle. The following results can be found in [6].

Theorem 1. *Given a nonnegative integer d. For each bivariate polynomial $p_d : \mathbb{R}^2 \to \mathbb{R}$ of degree d there exists a unique blossom (or polar form) of p_d $\mathcal{B}[p_d] : (\mathbb{R}^2)^d \to \mathbb{R}$ satisfying*

- *$\mathcal{B}[p_d]$ is symmetric,*

$$\mathcal{B}[p_d](z_1,\dots,z_d) = \mathcal{B}[p_d](z_{\pi(1)},\dots,z_{\pi(d)})$$

for any permutation π of the integers $1,\dots,d$.

- *$\mathcal{B}[p_d]$ is multiaffine,*

$$\mathcal{B}[p_d](z_1,(a\widehat{z}+b\widetilde{z}),z_3,\dots,z_d) = a\mathcal{B}[p_d](z_1,\widehat{z},z_3,\dots,z_d) + b\mathcal{B}[p_d](z_1,\widetilde{z},z_3,\dots,z_d),$$

where $a+b=1$.

- *$\mathcal{B}[p_d]$ is diagonal, $p_d(z) = \mathcal{B}[p_d](\underbrace{z,\dots,z}_{d})$, for all $z \in \mathbb{R}^2$.*

Define $\mathbb{P}_d$ as the space of bivariate polynomials of total degree d. Then, we have the following result which can be proved by using Theorem 1.

Lemma 2. *Let R_1, R_2 be two polynomials in $\mathbb{P}_1$. If $p(x,y) = R_1(x,y)R_2(x,y)$, then we have*

$$\mathcal{B}[p](z_1,z_2) = \frac{1}{2}(R_1(z_1)R_2(z_2) + R_1(z_2)R_2(z_1)).$$

2.2. Polynomials on triangles

Consider a triangle $\mathcal{T}(V_1,V_2,V_3)$ in a plane with vertices V_i, $i=1,2,3$. Then each polynomial $p \in \mathbb{P}_d$ on $\mathcal{T}$ has a unique representation

$$p(x,y) = \sum_{|\alpha|=d} b_\alpha \mathfrak{B}_\alpha^d(\lambda),$$

with $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}^3$ a multiindex of length $|\alpha| = \alpha_1+\alpha_2+\alpha_3$, $\lambda = (\lambda_1,\lambda_2,\lambda_3)$ the barycentric coordinates of (x,y) with respect to $\mathcal{T}$, and

$$\mathfrak{B}_\alpha^d(\lambda) = \frac{d!}{\alpha_1!\alpha_2!\alpha_3!}\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}, \tag{1}$$

the Bernstein Bézier polynomials of degree d on the triangle. The coefficients b_α are called the Bézier ordinates. The coefficients b_α are called Bézier ordinates, and the Bézier domain points ξ_α are defined as the points with barycentric coordinates $(\alpha_1/d, \alpha_2/d, \alpha_3/d)$. The points

$$(\xi_\alpha, b_\alpha) \in \mathbb{R}^3, \quad |\alpha| = d$$

are the Bézier control points of p.

On the other hand, Ramshaw [6] has shown that the Bézier ordinates for a polynomial relative to a triangle $\mathcal{T}$ can be obtained by evaluating the polynomial's blossom at the vertices of $\mathcal{T}$. More precisely, for all $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}^3$ where $|\alpha| = \alpha_1+\alpha_2+\alpha_3 = d$, we have

$$b_\alpha = \mathcal{B}[p_d]\left(\overbrace{\underbrace{V_1,\dots,V_1}_{\alpha_1},\underbrace{V_2,\dots,V_2}_{\alpha_2},\underbrace{V_3,\dots,V_3}_{\alpha_3}}^{d}\right).$$