Applied Mathematics and Computation 219 (2013) 10538–10547

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Some complete monotonicity properties for the (p,q)-gamma function

V.B. Krasniqi [a,*], H.M. Srivastava [b], S.S. Dragomir [c,d]

[a] Department of Mathematics and Computer Sciences, University of Prishtina, Avenue "Mother Tereza" Nr. 5, Prishtinë 10000, Republic of Kosova
[b] Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada
[c] Section of Mathematics, School of Engineering and Science, Victoria University of Technology, P.O. Box 14428, Melbourne City 8001, Australia
[d] School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa

ARTICLE INFO

Keywords:
Completely monotonic functions
Logarithmically completely monotonic functions
Log-convex functions
(p,q)-Gamma function
(p,q)-Psi function
Borel measure
Laplace transforms
Young's inequality

ABSTRACT

For the $\Gamma_{p,q}$-function, we derive several properties and characteristics related to convexity, log-convexity and complete monotonicity. Similar properties and characteristics of the corresponding (p,q)-analogue $\psi_{p,q}(x)$ of the digamma or the ψ-function have also been established. By applying the main results in this paper when $p \to \infty$ and $q \to 1$, we obtain all of the results given in several earlier works by (for example) Krasniqi, Shabani, and other authors. Some potential areas of applications of the results presented in this paper are also indicated.

© 2013 Elsevier Inc. All rights reserved.

1. Introduction

The familiar (Euler) gamma function $\Gamma(x)$ is defined (for $x > 0$) by

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t} dt \quad (x > 0).$$

The digamma (or psi-) function $\psi(x)$ is defined (for $x > 0$) as the logarithmic derivative of Euler's gamma function $\Gamma(x)$, that is, by

$$\psi(x) = \frac{d}{dx}\{\log \Gamma(x)\} = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The following integral and series representations are known (see [1,18,23]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n=1}^\infty \frac{x}{n(n+x)} \quad (x > 0), \tag{1.1}$$

where γ denotes the Euler–Mascheroni constant defined by (see also a recent work [7])

$$\gamma = -\psi(1) := \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \log n\right) \cong 0.57721\,56649\,01532\,86060\,65120\,90082\,40243\,1042\cdots. \tag{1.2}$$

* Corresponding author.
E-mail addresses: vali.99@hotmail.com (V.B. Krasniqi), harimsri@math.uvic.ca (H.M. Srivastava), sever.dragomir@vu.edu.au (S.S. Dragomir).

0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2013.04.034

Euler also introduced the following interesting variant of the gamma function $\Gamma(x)$ (see [3,21]):

$$\Gamma_p(x) = \frac{p!\, p^x}{x(x+1)\ldots(x+p)} = \frac{p^x}{x(1+\frac{x}{1})\ldots(1+\frac{x}{p})} \tag{1.3}$$

$$(x > 0;\ p \in \mathbb{N} := \{1,2,3,\ldots\} = \mathbb{N}_0 \setminus \{0\}),$$

so that

$$\Gamma(x) = \lim_{p\to\infty} \Gamma_p(x). \tag{1.4}$$

The p-analogue of the ψ-function is defined as the logarithmic derivative of the Γ_p-function as follows (see [16]):

$$\psi_p(x) = \frac{d}{dx}\{\log \Gamma_p(x)\} = \frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{1.5}$$

The following representations for the functions $\Gamma_p(x)$ and $\psi_p(x)$ hold true:

$$\Gamma_p(x) = \int_0^p \left(1 - \frac{t}{p}\right)^p t^{x-1} dt, \tag{1.6}$$

$$\psi_p(x) = \log p - \int_0^\infty \frac{e^{-xt}\left(1 - e^{-(p+1)t}\right)}{1 - e^{-t}} dt \tag{1.7}$$

and

$$\psi_p^{(m)}(x) = (-1)^{m+1} \cdot \int_0^\infty \frac{t^m\, e^{-xt}}{1 - e^{-t}} \left(1 - e^{-(p+1)t}\right) dt. \tag{1.8}$$

Jackson [9–11] (see also [23,25]) defined the basic (or q-) analogue of the gamma function as follows:

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \quad (0 < q < 1) \tag{1.9}$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x} q^{\binom{x}{2}} \quad (q > 1), \tag{1.10}$$

where, and in what follows,

$$(a;q)_\lambda = \frac{(a;q)_\infty}{(aq^\lambda;q)_\infty} \quad \text{and} \quad (a;q)_\infty = \prod_{j=0}^\infty (1 - aq^j)$$

and

$$[\lambda] := \frac{1-q^\lambda}{1-q}, \quad [0]! := 1 \quad \text{and} \quad [n]! := [1][2][3]\ldots[n] \quad (n \in \mathbb{N}).$$

The basic (or q-) gamma function $\Gamma_q(x)$ has the following integral representation (see, for details, [9–11]):

$$\Gamma_q(t) = \int_0^\infty x^{t-1} E_q^{-qx} d_q x,$$

where E_q^x defined by

$$E_q^x := \sum_{j=0}^\infty q^{\frac{j(j-1)}{2}} \frac{x^j}{[j]!} = [(1 + (1-q)x;q)]_\infty$$

is the q-analogue of the classical exponential function e^x. The q-analogue of the ψ-function is defined (for $0 < q < 1$) as the logarithmic derivative of the q-gamma function, that is, by (see also [17])

$$\psi_q(x) = \frac{d}{dx}\{\log \Gamma_q(x)\} = \frac{\Gamma_q'(x)}{\Gamma_q(x)}.$$

Many properties of the q-gamma function were derived by Askey [4] (see also [23, p. 490 *et seq.*]).

It is well-known that

$$\lim_{q\to 1-} \Gamma_q(x) \to \Gamma(x) \quad \text{and} \quad \lim_{q\to 1-} \psi_q(x) \to \psi(x).$$

Download English Version:

https://daneshyari.com/en/article/4628956

Download Persian Version:

https://daneshyari.com/article/4628956

Daneshyari.com