Applied Mathematics and Computation 219 (2013) 6457–6465

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Existence of solutions bounded together with their first derivatives of some systems of nonlinear functional differential equations

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ARTICLE INFO

Keywords:
Nonlinear system
Functional differential equation
Bounded continuously differentiable solutions
Iterated deviations

ABSTRACT

We give sufficient conditions for the system of nonlinear functional differential equations of the form

$$x'(t+1) = Ax'(t) + F(t, x(t), x(v^{(1)}(t)), \dots, x(v^{(k)}(t)), x'(g_1(t)), \dots, x'(g_l(t))),$$

where A is a matrix, $det\, A \neq 0$, $v^{(j)}(t) = \varphi_{j1}(t, x(\varphi_{j2}(\dots x(\varphi_{jm_j}(t, x(t)))\dots)))$, $j = \overline{1,k}$, and functions $F : \mathbb{R}_+ \times (\mathbb{R}^n)^{k+l+1} \to \mathbb{R}^n$, $g_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = \overline{1,l}$, $\varphi_{ji} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, $j = \overline{1,k}$, $i = \overline{1,m_j}$, are continuous, for the unique existence of continuously differentiable solutions $x(t)$ which are bounded together with their first derivatives on $\mathbb{R}_+ = [0,\infty)$ and satisfy the condition $\lim_{t\to+\infty} |x(t+1) - Ax(t)| = 0$.

1. Introduction and preliminaries

Particular cases of the following system of functional differential equations

$$x'(t+1) = Ax'(t) + F(t, x(t), x(f(t, x(t))), x'(g(t))), \tag{1}$$

where $t \in \mathbb{R}_+ = [0,\infty)$, A is an $n \times n$ matrix such that $det\, A \neq 0$, $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$, have been studied, e.g. in [1,2,12,14–17,19,23,24,38,39]. System (1) belongs to the class of differential equations partially solved with respect to the highest-order derivatives, which have been studied, for example, in [1–25,38,39].

Iteration methods in [26–30] motivated us to propose studying equations with continuous arguments, whose deviations of an argument depend on an unknown function which depend also of the function and so on (for related ideas see also [31–33]). We call them *iterated deviations*. They are introduced in [37], for the case of functional equations. For the case of non-linear functional differential equations see [34,35], and for the case of systems of nonlinear functional difference equations see [36].

Let, us usual, $C^1(\mathbb{R}_+)$ denote the space of continuously differentiable functions on $\mathbb{R}_+$. The space $BC^1(\mathbb{R}_+)$ is defined by

$$BC^1(\mathbb{R}_+) := \{x \in C^1(\mathbb{R}_+) : \|x\|_\infty < \infty\},$$

where the norm $\|x\|_\infty$ is defined by

$$\|x\|_\infty = \max\left\{\sup_{t\in\mathbb{R}_+} |x(t)|, \sup_{t\in\mathbb{R}_+} |x'(t)|\right\}.$$

So, the space $BC^1(\mathbb{R}_+)$ consists of all $C^1(\mathbb{R}_+)$ functions which are bounded together with their first derivatives on $\mathbb{R}_+$.

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http://dx.doi.org/10.1016/j.amc.2012.11.085

We give some sufficient conditions for the unique existence of a $BC^1(\mathbb{R}_+)$ solution of the next system of nonlinear functional differential equations

$$x'(t+1) = Ax'(t) + F(t, x(t), x(v^{(1)}(t)), \ldots, x(v^{(k)}(t)), x'(g_1(t)), \ldots, x'(g_l(t))), \tag{2}$$

where

$$v^{(j)}(t) = \varphi_{j1}(t, x(\varphi_{j2}(\ldots x(\varphi_{jm_j}(t, x(t)))\ldots))), \quad j = \overline{1,k},$$

A is a nonsingular $n \times n$ matrix, $F : \mathbb{R}_+ \times (\mathbb{R}^n)^{k+l+1} \to \mathbb{R}^n$, $g_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = \overline{1,l}$, $\varphi_{ji} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, $j = \overline{1,k}$, $i = \overline{1,m_j}$, are continuous functions, such that

$$\lim_{t \to +\infty} |x(t+1) - Ax(t)| = 0, \tag{3}$$

where $|x| = \max_{1 \leqslant i \leqslant n} |x_i|$.

Throughout the paper the symbol $|\cdot|$ denotes the modulus of a number, or a norm of a vector in $\mathbb{R}^n$, or a norm of a matrix, depending of the quantity appearing under the symbol.

Related results can be found, for example, in [3,8,13,15,18–20,22,38], some of which are considerably extended in this paper.

We finish this section with the following lemma which will be used in the proof of a main result in this paper.

Lemma 1. *Assume that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two sequences of nonnegative numbers, and that sequence $(x_n)_{n\in\mathbb{N}}$ satisfies the inequality*

$$x_n \leqslant a_n + b_n x_{n+1}, \quad n \in \mathbb{N}. \tag{4}$$

Then

$$x_1 \leqslant \sum_{j=1}^{k-1} a_j \prod_{i=1}^{j-1} b_i + x_k \prod_{i=1}^{k-1} b_i, \tag{5}$$

for every $k \in \mathbb{N}$.

Proof. We prove the lemma by the method of induction. For $k = 1$ inequality (5) trivially holds (actually, it becomes the equality $x_1 = x_1$). Assume that inequality (5) holds for some $k \in \mathbb{N}$. Using inequality (4) with $n = k$ in (5), and the fact that the number $\prod_{i=1}^{k-1} b_i$ is nonnegative, we obtain

$$x_1 \leqslant \sum_{j=1}^{k-1} a_j \prod_{i=1}^{j-1} b_i + x_k \prod_{i=1}^{k-1} b_i \leqslant \sum_{j=1}^{k-1} a_j \prod_{i=1}^{j-1} b_i + (a_k + b_k x_{k+1}) \prod_{i=1}^{k-1} b_i = \sum_{j=1}^{k} a_j \prod_{i=1}^{j-1} b_i + x_{k+1} \prod_{i=1}^{k} b_i, \tag{6}$$

finishing the inductive proof of the lemma. □

2. Main results

This section we begin with listing conditions which we assume in the rest of this paper.

(i) *det* $A \neq 0$;

(ii) the vector function $F(t, x, \vec{y}, \vec{z})$ is continuous for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $\vec{y} \in (\mathbb{R}^n)^k$, $\vec{z} \in (\mathbb{R}^n)^l$,

$$F(t, 0, \vec{0}_{(\mathbb{R}^n)^k}, \vec{0}_{(\mathbb{R}^n)^l}) \equiv 0 \tag{7}$$

and

$$|F(t, x, \vec{y}, \vec{z}) - F(t, u, \vec{v}, \vec{w})| \leqslant \eta_0(t)|x - u| + \sum_{j=1}^{k} \eta_j(t)|y_j - v_j| + \sum_{i=1}^{l} \eta_{k+i}(t)|z_i - w_i|, \tag{8}$$

where $\eta_j(t)$, $j = \overline{0, k+l}$, are continuous and nonnegative functions for $t \in \mathbb{R}_+$, and $x, u \in \mathbb{R}^n$, $\vec{y}, \vec{v} \in (\mathbb{R}^n)^k$, $\vec{z}, \vec{w} \in (\mathbb{R}^n)^l$;

(iii) functions $g_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = \overline{1,l}$, and $\varphi_{ji} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, $j = \overline{1,k}$, $i = \overline{1,m_j}$, are continuous and

$$|\varphi_{ji}(t, x) - \varphi_{ji}(t, y)| \leqslant l_{ji}|x - y|, \tag{9}$$

where l_{ji}, $j = \overline{1,k}$, $i = \overline{1,m_j}$, are some non-negative constants, $t \in \mathbb{R}_+$, and $x, y \in \mathbb{R}^n$;