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On the spectral norms of the matrices connected to integer number sequences

Durmuş Bozkurt

Department of Mathematics, Science Faculty of Selçuk University, Turkey

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ABSTRACT

In this paper, we compute the spectral norms of the matrices related with integer sequences and we give two examples related with Fibonacci, Lucas, Pell and Perrin numbers.

1. Introduction

In [1], the upper and lower bounds for the spectral norms of r-circulant matrices are obtained by Shen and Cen. The lower bounds for the norms of Cauchy–Toeplitz and Cauchy–Hankel matrices are given by Wu in [2]. In [3–5], Solak and Bozkurt have found some bounds for the norms of Cauchy–Toeplitz, Cauchy–Hankel and circulant matrices.

Let A be any $n \times n$ complex matrix. The well known spectral norm of the matrix A is

$$\|A\|_2 = \sqrt{\max_{1 \leqslant i \leqslant n} |\lambda_i(A^H A)|},$$

where $\lambda_i(A^H A)$ is eigenvalue of $A^H A$ and A^H is conjugate transpose of the matrix A. k-principal minor of the matrix A is denoted by

$$A\begin{pmatrix} i_1 i_2 \dots i_k \\ i_1 i_2 \dots i_k \end{pmatrix} = \begin{vmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \dots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \dots & a_{i_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k,i_1} & a_{i_k,i_2} & \dots & a_{i_k,i_k} \end{vmatrix}, \tag{1}$$

where $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n \ (1 \leqslant k \leqslant n)$ [6].

By a circulant matrix of order n is meant a square matrix of the form [8]

$$C = circ(c_0, c_1, \dots, c_{n-1}) = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{bmatrix}.$$

E-mail address: dbozkurt@selcuk.edu.tr

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Let $A = [a_{ij}]$ be an $n \times n$ positive matrix: $a_{ij} > 0$ for $1 \leqslant i,\ j \leqslant n$. Then there is a positive real number r, called the Perron root or the Perron–Frobenius eigenvalue, such that r is an eigenvalue of A and any other eigenvalue λ (possibly, complex) is strictly smaller than r in absolute value, $|\lambda| < r$. Thus, the spectral radius $\rho(A)$ is equal to r [10, p. 64].

Now we define our matrices. x_is are any integer numbers sequence for $i = 1, 2, \ldots, n$. Let matrix A_x be following form:

$$A_x = [a_{ij}]_{i,j=1}^{n} = [x_i - x_j]_{i,j=1}^{n}. \tag{2}$$

Obviously, A_x is skew-symmetric matrix, i.e. $A_x^T = -A_x$. Since eigenvalues of a skew-hermitian matrix are pure imaginary, eigenvalues of the matrix iA_x are real where i is complex unity.

(y_n) is any positive integer numbers sequence and y_i is ith the component of the sequence (y_n) for $i = 0, 1, 2, \ldots$. Let matrix C_y be following form:

$$C_y = circ(y_0, y_1, \ldots, y_{n-1}). \tag{3}$$

The main objective of this paper is to obtain the spectral norms of the matrices A_x and C_y in (2) and (3).

2. Main results

Theorem 1. *Let the matrices A_x be as in (2). Then*

$$\|A_x\|_2 = \left[\sum_{1 \leqslant r < s \leqslant n} (x_r - x_s)^2\right]^{1/2}, \tag{4}$$

where $n \geqslant 4$.

Proof. If we substract $(i-1)$th row from ith row of the matrix A_x for $i = n, n-1, \ldots, 2$, then we obtain

$$B_x = \begin{bmatrix} 0 & x_1 - x_2 & x_1 - x_3 & \ldots & x_1 - x_n \\ x_2 - x_1 & x_2 - x_1 & x_2 - x_1 & \ldots & x_2 - x_1 \\ x_3 - x_2 & x_3 - x_2 & x_3 - x_2 & \ldots & x_3 - x_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} - x_{n-2} & x_{n-1} - x_{n-2} & x_{n-1} - x_{n-2} & \ldots & x_{n-1} - x_{n-2} \\ x_n - x_{n-1} & x_n - x_{n-1} & x_n - x_{n-1} & \ldots & x_n - x_{n-1} \end{bmatrix}.$$

Obviously, $rank(B_x) = rank(A_x) = 2$. Since the matrix A_x is skew-symmetric, the matrix iA_x is symmetric where i is complex unity. Then all the eigenvalues of the matrix iA_x are real numbers. Moreover, $rank(A_x) = rank(iA_x)$. Since determinants of all k-square submatrices of the matrix iA_x are zero for $k \geqslant 3$, all principal k-minors of the matrix iA_x are zero for $k \geqslant 3$. Then characteristic polynomial of the matrix iA_x

$$\Delta_{iA_x}(\lambda) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2}, \tag{5}$$

where $(-1)^k a_k$ is the sum of principal k-minors of the matrix iA_x where $k = 1, 2$ [9]. On the other hand since $rank(iA_x) = 2$, two eigenvalues of the matrix iA_x are nonzero. If $i\lambda$ is the eigenvalue of the matrix A_x, then $-i\lambda$ is an eigenvalue of A_x. Then $a_1 = -tr(A_x) = -tr(iA_x) = \sum_{k=1}^{n} \lambda_k = 0$ where λ_k are the eigenvalues of the matrix iA_x. Coefficient a_2 is the sum of principal 2-minors of any square matrix A_x. i.e.

$$a_2 = \sum\nolimits_{1 \leqslant r < s \leqslant n} A\begin{pmatrix} r & s \\ r & s \end{pmatrix}.$$

Then we have

$$a_2 = \sum_{1 \leqslant r < s \leqslant n} iA_x\begin{pmatrix} r & s \\ r & s \end{pmatrix} = \sum_{1 \leqslant r < s \leqslant n} \begin{vmatrix} i(x_r - x_r) & i(x_r - x_s) \\ i(x_s - x_r) & i(x_s - x_s) \end{vmatrix} = \sum_{1 \leqslant r < s \leqslant n} \begin{vmatrix} 0 & i(x_r - x_s) \\ -i(x_r - x_s) & 0 \end{vmatrix} = -\sum_{1 \leqslant r < s \leqslant n} (x_r - x_s)^2.$$

Hence from (5) we obtain

$$\Delta_{iA_x}(\lambda) = \lambda^n - \left(\sum_{1 \leqslant r < s \leqslant n} (x_r - x_s)^2\right) \lambda^{n-2}.$$

Then

$$\|iA_x\|_2^2 = \|A_x\|_2^2 = \sum_{1 \leqslant r < s \leqslant n} (x_r - x_s)^2.$$

The proof of (4) is completed. □