Applied Mathematics and Computation 219 (2013) 6605–6617

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Hopf bifurcation analysis of a system of coupled delayed-differential equations

C. Çelik [a], H. Merdan [b,*]

[a] Bahcesehir University, Faculty of Science and Letters, Department of Mathematics–Computers, Besiktas, Istanbul, Turkey
[b] TOBB University of Economics and Technology, Faculty of Science and Letters, Department of Mathematics, Söğütözü Cad. No. 43, 06560 Ankara, Turkey

ARTICLE INFO

Keywords:
Hopf bifurcation
Delay differential equation
Time delay
Stability
Periodic solutions

ABSTRACT

In this paper, we have considered a system of delay differential equations. The system without delayed arises in the Lengyel–Epstein model. Its dynamics are studied in terms of local analysis and Hopf bifurcation analysis. Linear stability is investigated and existence of Hopf bifurcation is demonstrated via analyzing the associated characteristic equation. For the Hopf bifurcation analysis, the delay parameter is chosen as a bifurcation parameter. The stability of the bifurcating periodic solutions is determined by using the center manifold theorem and the normal form theory introduced by Hassard et al. (1981) [7]. Furthermore, the direction of the bifurcation, the stability and the period of periodic solutions are given. Finally, the theoretical results are supported by some numerical simulations.

1. Introduction

Recently, the ordinary and partial differential equations models involving delays have attracted the attention of great number of investigators and were successfully applied to problems in the different research areas ranging from biology to economics [1–8]. In order to reflect that the dynamical behaviors of models that depend on the past history of the system, it is often necessary to incorporate time–delays into the models [1–8]. The models involving time delays have been widely studied in fields as diverse as biology, population dynamics, neural networks, feedback controlled mechanical systems, machine tool vibrations and lasers. Delay effects can also be exploited to control nonlinear systems ([2,6,8–12,14,15,17–19,21] and the references therein).

It is well known that the studies on dynamical systems not only include a discussion of stability, attractivity and persistence, but also include many dynamical behaviors such as periodic phenomenon, bifurcation and chaos [2,5–8]. In delay differential equations, periodic solutions can arise through the (local) Hopf bifurcation. Several methods for analyzing the nature of Hopf bifurcations have been described in the literature. Integral averaging has been used by Chow and Mallet-Paret, the Fredholm alternative has been used by Iooss and Joseph, the Implicit Function Theorem has been used by Hale and Lunel, multi-scale expansion has been used by Nayfeh et al., and center manifold projection has been used by Hassard et al. and Stépán and Kalmár-Nagy ([2] and see the references therein). Center manifold theory is one of the rigorous mathematical tools to study bifurcations of delay differential equations [7].

In this paper, we consider the following system of coupled delayed-differential equations

$$\begin{cases} u' = a - u - 4\frac{uv(t-\tau)}{1+u^2} \\ v' = \sigma b\left(u - 4\frac{uv(t-\tau)}{1+u^2}\right) \end{cases}, \tag{1}$$

* Corresponding author.
E-mail addresses: canancelik@bahcesehir.edu.tr (C. Çelik), merdan@etu.edu.tr (H. Merdan).

0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2012.12.063

where τ is the delay parameter that is always nonnegative. When $\tau = 0$ in Eq. (1), the system reduces to the ODE part of the Lengyel–Epstein model which is a system of coupled reaction–diffusion equations that is used for the CIMA reaction (see [4] and the references therein). In the content of the CIMA reaction, $u(t)$ and $v(t)$ denote the chemical concentrations of the activator iodine (I^-) and the inhibitor chlorite (ClO_2^-), respectively, at time t. The positive parameters a and b are related to the feed concentration; similarly, the positive parameter σ is a resealing parameter depending on the concentration of the starch. For the case $\tau = 0$ in Eq. (1), Yi et al. [20] has studied a detailed Hopf bifurcation analysis for this ODE model (and also the associated PDE model, too) by choosing b as the bifurcation parameter and derive conditions on the parameters for determining the direction and the stability of the bifurcating periodic solution.

Our aim in this paper is to study a detailed Hopf bifurcation analysis of Eq. (1) by choosing the delay parameter τ as a bifurcation parameter. We investigate the linear stability and the existence of Hopf bifurcation by analyzing the associated characteristic equation and determine the required conditions on parameters. In other words, we use the Hopf bifurcation theorem in [7] to investigate the effect of delay on solutions of Eq. (1) and to show that when τ passes through a certain critical value, the positive equilibrium loses its stability and a Hopf bifurcation occurs. Furthermore, when τ takes a sequence of critical values, the system (1) undergoes a Hopf bifurcation near positive equilibrium at these critical values of τ. Beside, we determine the direction of the bifurcation, and the stability and the period of the bifurcating periodic solution by using the center manifold theorem and the normal form theory introduced by Hassard et al. [7]. Finally, we give some numerical simulations in order to support the theoretical results obtained.

This paper is organized as follows. In Section 2, the stability analysis of constant equilibrium points is investigated and the existence of Hopf bifurcation is showed. In Section 3, bifurcation properties including direction, stability and period of the bifurcating periodic solutions are studied. Finally, in Section 4, we give numerical examples by using MATLAB programing to show the effect of time delay and support our theoretical results.

2. Stability analysis and Hopf bifurcation

First, let us check the stability of the equilibrium points of the model (1). The system has a unique positive equilibrium point $(u^*, v^*) = (\alpha, 1 + \alpha^2)$ where $\alpha = \frac{a}{5}$. Notice that Eq. (1) without delay has also the same equilibrium point, i.e., the delay does not change equilibria. By shifting the equilibrium point (u^*, v^*) to the origin via the transformations $x(t) = u(t) - u^*$ and $y(t) = v(t) - v^*$, and then linearizing the new system around zero, we get the following

$$\begin{cases} x' = \left(\frac{3\alpha^2-5}{1+\alpha^2}\right)x(t) + \left(\frac{-4\alpha}{1+\alpha^2}\right)y(t-\tau) + f(x(t), y(t-\tau)) + h.o.t. \\ y' = \left(\frac{2\sigma b\alpha^2}{1+\alpha^2}\right)x(t) + \left(\frac{-\sigma b\alpha}{1+\alpha^2}\right)y(t-\tau) + g(x(t), y(t-\tau)) + h.o.t. \end{cases}, \tag{2}$$

where

$$f(x(t), y(t-\tau)) := \frac{1}{2}\left(\frac{24\alpha - 8\alpha^3}{(1+\alpha^2)^2}x(t) + \frac{-8+8\alpha^2}{(1+\alpha^2)^2}x(t)y(t-\tau)\right), \tag{3}$$

$$g(x(t), y(t-\tau)) := \frac{1}{2}\left(\frac{6\sigma b\alpha - 2\sigma b\alpha^3}{(1+\alpha^2)^2}x(t) + \frac{\sigma b\alpha^3 - 2\sigma b}{(1+\alpha^2)^2}x(t)y(t-\tau)\right) \tag{4}$$

and $h.o.t.$ denotes the higher order terms. Now, using the transformation $t = \tau s$ one obtains

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \tau J\begin{pmatrix} x(t) \\ y(t-1) \end{pmatrix} + \tau\begin{pmatrix} F(x(t), y(t-1)) \\ G(x(t), y(t-1)) \end{pmatrix} + h.o.t., \tag{5}$$

where the Jacobian matrix J is

$$J = \begin{pmatrix} \frac{3\alpha^2-5}{1+\alpha^2} & \frac{-4\alpha}{1+\alpha^2} \\ \frac{2\sigma b\alpha^2}{1+\alpha^2} & \frac{-\sigma b\alpha}{1+\alpha^2} \end{pmatrix}, \tag{6}$$

$$F(x(t), y(t-1)) := \frac{1}{2}\left(\frac{24\alpha - 8\alpha^3}{(1+\alpha^2)^2}x(t) + \frac{-8+8\alpha^2}{(1+\alpha^2)^2}x(t)y(t-1)\right) \tag{7}$$

and

$$G(x(t), y(t-1)) := \frac{1}{2}\left(\frac{6\sigma b\alpha - 2\sigma b\alpha^3}{(1+\alpha^2)^2}x(t) + \frac{\sigma b\alpha^3 - 2\sigma b}{(1+\alpha^2)^2}x(t)y(t-1)\right). \tag{8}$$

Here, for convenience, the symbol t is again used for the new independent variable (instead of s). It is easy to see that the characteristic equation is

$$\lambda^2 - tr(J)\lambda - \det(J) = 0, \tag{9}$$