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# Membership tests for images of algebraic sets by linear projections

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ABSTRACT

Given a witness set for an irreducible variety $V$ and a linear map $\pi$, we describe membership tests for both the constructible algebraic set $\pi(V)$ and the algebraic set $\overline{\pi(V)}$. We also provide applications and examples of these new tests including computing the codimension one components of $\overline{\overline{\pi(V)} \setminus \pi(V)}$. Additionally, we also describe computing the geometric genus of a curve section of an irreducible component of the solution set of a polynomial system and a test for deciding whether a plane quartic curve is a Lüroth quartic.



## 1. Introduction

Given a polynomial system $f : \mathbb{C}^N \to \mathbb{C}^n$, an $\ell$-dimensional irreducible component $V \subset f^{-1}(0)$, and a linear map $\pi : \mathbb{C}^N \to \mathbb{C}^K$, a "witness set" for $\overline{\pi(V)}$ was constructed in [7] from a witness set for $V$, hereafter called a *pseudo-witness set* for $\overline{\pi(V)}$. This approach reduces computations on $\overline{\pi(V)}$ to computations on $V$ without using elimination theory to construct a polynomial system $g$ such that $\overline{\pi(V)}$ is an irreducible component of $g^{-1}(0)$.

The main results of this article, presented in Section 3, are algorithms for performing a numerical membership test for both $\pi(V)$ and $\overline{\pi(V)}$.

Chevalley's theorem [4] states that the image of a constructible set, e.g., $\pi(V)$, is a constructible set.[3] Effective symbolic methods for performing computations with constructible sets are discussed in [5,19].

In Section 4, we use these membership tests to compute a numerical decomposition of the irreducible components of $\overline{\overline{\pi(V)} \setminus \pi(V)}$ of codimension one in $\overline{\pi(V)}$ and use this to develop an approach for computing the geometric genus of a generic curve section of $\overline{\pi(V)}$.

The necessary background material is presented in Section 2 which also codifies the properties of our substitute for witness sets into the notion of a pseudo-witness set.

In Section 5, we present examples using our new membership tests.

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[2] This author was supported by the Duncan Chair of the University of Notre Dame, Institute Mittag-Leffler (Djursholm, Sweden), and NSF Grant DMS-0712910.
[3] A *constructible subset* of an algebraic set $X$ is any set in the Boolean algebra of subsets of $X$ obtained by starting with algebraic subsets of $X$ and closing up under the operations of finite unions and complementation.

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## 2. Background material

We collect some background material in this section. Throughout, we assume $f : \mathbb{C}^N \to \mathbb{C}^n$ is a polynomial system and define $\mathcal{V}(f)$ to be the set of points in $\mathbb{C}^N$ which $f$ maps to 0. The algebraic set $\mathcal{V}(f)$ is reduced and, in particular, all of the irreducible components of $\mathcal{V}(f)$ have multiplicity one. We let $f^{-1}(0)$ denote $\mathcal{V}(f)$ with its underlying scheme structure, which includes the multiplicity information of the components of $\mathcal{V}(f)$ with regard to $f$.

### 2.1. Witness sets

Suppose that $V \subset f^{-1}(0)$ is an $\ell$-dimensional irreducible algebraic set of degree $d$. A *witness set* for $V$ is the triple $\{f, \mathcal{L}, W\}$ where $\mathcal{L}$ consists of $\ell$ general linear polynomials on $\mathbb{C}^N$ and $W = V \cap \mathcal{V}(\mathcal{L})$. The *witness point set* $W$ consists of $d$ points. A finite set $\mathcal{W}$ with $W \subset \mathcal{W} \subset V$ is called a *witness point superset* for $V$.

The *multiplicity* of $V$ with respect to $f$ is the multiplicity of any $w \in W$ as a root of $\begin{bmatrix} f \\ \mathcal{L} \end{bmatrix}$. The component $V$ is said to be *generically reduced* with respect to $f$ if the multiplicity of $V$ with respect to $f$ is 1. Otherwise, $V$ is said to be *generically nonreduced*, which we consider in the following section. See Chap. 13 [23] for more details regarding witness sets.

### 2.2. Deflation

If $V$ is generically nonreduced with respect to $f$, then the deflation approach of [10] produces a polynomial system $F : \mathbb{C}^N \to \mathbb{C}^m$, with $m \geqslant n$, such that $F^{-1}(0)$ has an irreducible and generically reduced component $\widehat{V}$ which, as a set, is equal to $V$. By renaming as necessary, we will assume *without loss of generality* that $V$ is generically reduced with respect to $f$.

It should be noted that more traditional versions of deflation (see also [8,11,12] and Section 13.3.2 and 15.2.2 [23]) change the dimension of the ambient space and may replace $V$ with an algebraic set $V'$ that maps generically one-to-one onto a dense subset of $V$.

### 2.3. Randomization

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system and $1 \leqslant k \leqslant n$. For $A \in \mathbb{C}^{k\times(n-k)}$, let

$$\mathcal{R}(f;k) = [I_k A] \cdot f,$$

where $I_k$ is the $k \times k$ identity matrix. It is a consequence of Bertini's theorem, e.g. [22] or Section 13.5 [23], that any irreducible codimension $k$ component of $\mathcal{V}(f)$ is an irreducible component of $\mathcal{V}(\mathcal{R}(f;k))$ for a nonempty Zariski open (and hence dense) set of matrices $A \in \mathbb{C}^{k\times(n-k)}$. Thus, we will assume *without loss of generality* that $f : \mathbb{C}^N \to \mathbb{C}^k$ is a polynomial system where $V \subset f^{-1}(0)$ is a codimension $k$ irreducible component.

### 2.4. Pseudo-witness sets

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system and $\{f, \mathcal{L}, W\}$ be a witness set for an irreducible and generically reduced component $V \subset f^{-1}(0)$ of dimension $\ell$. Suppose that $\pi : \mathbb{C}^N \to \mathbb{C}^K$ is a linear map and $B \in \mathbb{C}^{K\times N}$ such that $\pi(x) = Bx$.

Even though the set $\pi(V)$ might not be an algebraic set, it is very close to an algebraic set. More specifically, $\pi(V)$ is a *constructible algebraic set* which means that it is a member of the Boolean algebra of sets constructed from algebraic sets by the operations of finite unions, finite intersections, and complementation. A typical example is the projection onto $(x, y)$ of $\mathcal{V}(x - yz)$: the image is $\left(\mathbb{C}^2 \setminus \mathcal{V}(y)\right) \cup \{(0,0)\}$.

The closure of a constructible algebraic set $C$ in the complex topology $\overline{C}$ is the same as the closure of $C$ in the Zariski topology. The same statement holds for the interior $C^\circ$ of $C$ with $\overline{C^\circ} = \overline{C}$. In particular, since the dimensions of $\overline{C}$ and $C^\circ$ are equal, the dimension of $C$ is well-defined. Finally, if $\overline{C}$ is pure $k$-dimensional, then $\overline{C} \cap L = C^\circ \cap L$ for a general affine linear space $L$ of codimension $k$. Additional details for constructible algebraic sets is provided in Appendix A [23].

Let $\ell' = \dim \overline{\pi(V)}$. For $i = 1, \ldots, \ell'$, let $b_i \in \mathbb{C}^N$ be general elements in the row span of $B$ and, for $i = \ell' + 1, \ldots, \ell$, let $b_i \in \mathbb{C}^N$ be general elements in $\mathbb{C}^N$. We call the quadruple $\{f, \pi, \mathcal{L}', W'\}$ [7], where

$$\mathcal{L}'(x) = \begin{bmatrix} b_1 \cdot x - 1 \\ \vdots \\ b_\ell \cdot x - 1 \end{bmatrix} \quad \text{and} \quad W' = V \cap \mathcal{V}(\mathcal{L}'),$$

a *pseudo-witness set* for $\pi(V)$ with $\deg \overline{\pi(V)} = |\pi(W)|$.

A pseudo-witness set may be efficiently used to fulfill the same tasks for which a witness set for $\overline{\pi(V)}$ would be used if we had a set of polynomials on $\mathbb{C}^K$ whose solution set contained $\overline{\pi(V)}$ as an irreducible component. One example is using pseudo-witness sets in place of witness sets to work with the numerical irreducible decomposition [20] of the closure of the image of an algebraic map, e.g. Section 2.1.3 [1].