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Coefficient estimates for the inverses of a certain general class of spirallike functions

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ABSTRACT

In the present paper, the authors derive several sharp coefficient estimates for the function class $\left(\hat{S}_{\alpha}^{\beta}\right)^{-1}$ consisting of the inverses of functions in a certain class $\hat{S}_{\alpha}^{\beta}$ of spirallike functions in the open unit disk $\mathbb{U}$, which was introduced by Libera. They also obtain a number of sharp coefficient bounds of functions in the more general classes Σ_{α}^{β} and $\left(\Sigma_{\alpha}^{\beta}\right)^{-1}$, each of which is introduced here. Some of the results derived in this paper would generalize those in a recent work of Kapoor and Mishra.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

that is, normalized by

$$f(0) = f'(0) - 1 = 0,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

In our present investigation, the class of functions $g(z)$ of the form:

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \cdots, \tag{2}$$

which are analytic and univalent in

$$\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| > 1\},$$

will be denoted by Σ.

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Definition 1 (*see Robertson [20]*). Let the function $f \in \mathcal{A}$ $(0 \leqq \alpha < 1)$ be constrained by the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{3}$$

We then say that the function $f(z)$ is starlike of order α in $\mathbb{U}$. We denote by $\mathcal{S}^*(\alpha)$ the class of starlike functions of order α in $\mathbb{U}$.

It is known that (see, for example, [4]; see also [25])

$$\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) \equiv \mathcal{S}^* \subset \mathcal{S} \quad (0 \leqq \alpha < 1).$$

The class of starlike functions of order α $(0 \leqq \alpha < 1)$ in Δ is denoted by $\Sigma^*(\alpha)$, that is, a function $g \in \Sigma^*(\alpha)$ if and only if $g \in \Sigma$ satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \Delta;\ 0 \leqq \alpha < 1).$$

Spaček [23] extended the class of $\mathcal{S}^*$ by introducing the class of spirallike functions of type β in $\mathbb{U}$ and gave the following analytical characterization of spirallikeness functions of type β in $\mathbb{U}$.

Theorem A (see Spaček [23]). *Let $f \in \mathcal{A}$ and suppose that the parameter β is constrained by $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. Then $f(z)$ is a spirallike function of type β in $\mathbb{U}$ if and only if*

$$\Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > 0 \quad \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}\right). \tag{4}$$

We denote the class of spirallike functions of type β in $\mathbb{U}$ by $\hat{\mathcal{S}}_\beta$.

Libera [15] extended the classes $\mathcal{S}^*(\alpha)$ and $\hat{\mathcal{S}}_\beta$ by introducing the analytic function class $\hat{\mathcal{S}}_\alpha^\beta$ in $\mathbb{U}$ as follows.

Definition 2 (see Libera [15]). Let $f \in \mathcal{A}$. Suppose also that the parameters α and β are constrained by

$$0 \leqq \alpha < 1 \quad \text{and} \quad -\frac{\pi}{2} < \beta < \frac{\pi}{2}.$$

We then say that $f \in \hat{\mathcal{S}}_\alpha^\beta$ if and only if

$$\Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > \alpha\cos\beta \quad \left(z \in \mathbb{U};\ 0 \leqq \alpha < 1;\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}\right). \tag{5}$$

Obviously, we have

$$\hat{\mathcal{S}}_\alpha^0 = \mathcal{S}^*(\alpha) \quad \text{and} \quad \hat{\mathcal{S}}_0^\beta = \hat{\mathcal{S}}_\beta.$$

Definition 3. Let $g \in \Sigma$. Suppose also that the parameters α and β are constrained by

$$0 \leqq \alpha < 1 \quad \text{and} \quad -\frac{\pi}{2} < \beta < \frac{\pi}{2}.$$

We then say that $g \in \Sigma_\alpha^\beta$ if and only if

$$\Re\left(e^{i\beta}\frac{zg'(z)}{g(z)}\right) > \alpha\cos\beta \quad \left(z \in \Delta;\ 0 \leqq \alpha < 1;\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}\right). \tag{6}$$

We also have

$$\Sigma^*(\alpha) := \Sigma_\alpha^0 \quad (0 \leqq \alpha < 1).$$

Let $\mathcal{S}^{-1}$ be the class of the inverse functions f^{-1} of functions $f \in \mathcal{S}$ with the following Taylor–Maclaurin series expansion:

$$f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} A_n\omega^n, \tag{7}$$

in some disk $|\omega| < r_0(f)$ in the complex ω-plane. Suppose also that Σ^{-1} is the class of the inverse functions g^{-1} of the functions $g \in \Sigma$ with the following series expansion:

$$g^{-1}(\omega) = \omega + B_0 + \frac{B_1}{\omega} + \frac{B_2}{\omega^2} + \cdots, \tag{8}$$

in some neighborhood of the point at infinity in the complex ω-plane. The function classes

$$\left(\mathcal{S}^*(\alpha)\right)^{-1}, \quad \left(\Sigma^*(\alpha)\right)^{-1}, \quad \left(\hat{\mathcal{S}}_\alpha^\beta\right)^{-1} \quad \text{and} \quad \left(\Sigma_\alpha^\beta\right)^{-1},$$

are defined analogously.