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Complex dynamics of derivative-free methods for nonlinear equations ☆

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ABSTRACT

The dynamical behavior of two iterative derivative-free schemes, Steffensen and M4 methods, is studied in case of quadratic and cubic polynomials. The parameter plane is analyzed for both procedures on quadratic polynomials. Different dynamical planes are showed when the mentioned methods are applied on particular cubic polynomials with real or complex coefficients. The property of immersion of the basins of attraction in all cases is analyzed.

1. Introduction

The application of iterative methods for solving nonlinear equations $f(z)=0$, with $f:\mathbb{C}\rightarrow\mathbb{C}$, gives rise to rational functions whose dynamics are not well-known. There is an extensive literature on the study of iteration of rational mappings of a complex variable (see, for example, [1,2]). The simplest model is obtained when $f(z)$ is a quadratic polynomial and the iterative process is Newton's method. The dynamics of this iterative scheme has been widely studied (see, for instance, [2–4]).

The analysis of the dynamics of Newton's method has been extended to other point-to-point iterative methods, used for solving nonlinear equations with convergence higher than two (see, for example, [5–9]).

Most of the iterative methods analyzed from the dynamical point of view are schemes with derivatives in their iterative expressions. Unlike Newton's methods, the derivative-free scheme of Steffensen has been less studied. We can find some dynamical comments on this method in [5,10].

The efficiency index I performs a way to classify the iterative methods. This index is introduced by Ostrowski in [11] as $I=c^{1/d}$, where c is the convergence order of the method, and d the number of functional evaluations per iteration. Kung and Traub conjectured in [12] that $c\leqslant 2^{d-1}$. If the upper bound is reached, the method is considered optimal.

In this paper, we analyze the dynamics of two derivative-free iterative procedures, optimal in the sense of Kung–Traub's conjecture, of order two and four.

As it is well-known, if we replace the derivative of Newton's iterative expression by the progressive finite difference, we obtain Steffensen's method (see [13]), whose iterative expression is

$$z_{n+1}=z_n-\frac{f^2(z_n)}{f(v_n)-f(z_n)},\quad n=1,2,\ldots, \tag{1}$$

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where $v_n = z_n + f(z_n)$. As in Newton's method, Steffensen's one has quadratic convergence and the same efficiency index. From Kung–Traub's conjecture, Steffensen's method is optimal.

The fixed point operator of Steffensen's method on a polynomial $p(z)$ is

$$S_p(z) = z - \frac{p^2(z)}{p(v) - p(z)}, \tag{2}$$

where $v = z + p(z)$.

A common guideline used to improve the local order of convergence is the composition of two iterative methods, as shown in [13]. This technique obtains iterative schemes or order $c_1 \cdot c_2$, where c_1 and c_2 are the convergence order of the involved methods. The method M4 (see [14]) is obtained by composing Newton's and Steffensen's methods and using the Pade's approximant of degree one in order to avoid the last evaluation of the derivative. The iterative scheme is

$$\begin{aligned} y_n &= z_n - \frac{f^2(z_n)}{f(v_n) - f(z_n)}, \\ z_{n+1} &= y_n - \frac{f(y_n)f[z_n, v_n]}{f[z_n, y_n]f[y_n, v_n]}, \quad n = 1, 2, \ldots, \end{aligned} \tag{3}$$

where $f[x, y] = \frac{f(y)-f(x)}{y-x}$ is the divided difference of order one.

This method is fourth-order convergent and it is optimal from the point of view of the Kung–Traub's conjecture.

The fixed point operator of M4 on $p(z)$ is

$$M_p(z) = y - \frac{p(y)p[z, v]}{p[z, y]p[y, v]}, \tag{4}$$

where $v = z + p(z)$ and $y = z - \frac{f^2(z)}{f(v)-f(z)}$.

In order to study the dynamical behavior of an iterative method when is applied to a polynomial $p(z)$, it is necessary to recall some basic dynamical concepts. For a more extensive and comprehensive review of these concepts, see [15,16].

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function, where $\hat{\mathbb{C}}$ is the Riemann sphere. The orbit of a point $z_0 \in \hat{\mathbb{C}}$ is defined as

$$\{z_0, R(z_0), \ldots, R^n(z_0), \ldots\}.$$

The dynamical behavior of the orbit of a point on the complex plane can be classified depending on its asymptotic behavior. In this way, a point $z_0 \in \mathbb{C}$ is a fixed point of R if $R(z_0) = z_0$. A fixed point is attracting, repelling or neutral if $|R'(z_0)|$ is less than, greater than or equal to 1, respectively. Moreover, if $|R'(z_0)| = 0$, the fixed point is superattracting. The value of the derivative of the rational function at z_0, $|R'(z_0)|$, is called multiplier of this fixed point.

Let z_f^* be an attracting fixed point of the rational function R. The basin of attraction of an attracting fixed point $\mathcal{A}(z_f^*)$ is defined as the set of pre-images of any order such that

$$\mathcal{A}(z_f^*) = \left\{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to z_f^*, n \to \infty\right\}.$$

The set of points whose orbits tends to an attracting fixed point z_f^* is defined as the Fatou set, $\mathcal{F}(R)$. The complementary set, the Julia set $\mathcal{J}(R)$, is the closure of the set consisting of its repelling fixed points, and establishes the borders between the basins of attraction.

It is possible to find the fixed and critical points from the fixed point operator associated to each method, $O_p(z)$ on a polynomial $p(z)$. The fixed points z_f verify

$$O_p(z) = z, \tag{5}$$

and the critical points z_c validate

$$O'_p(z) = 0. \tag{6}$$

The attracting fixed points z_f^* are the points z_f such that

$$|O'_p(z_f)| < 1. \tag{7}$$

If $|O'_p(z_f)| = 0$, the fixed point is superattracting.

Mayer and Schleicher define in [17] the immediate basin of attraction of a superattracting fixed point z_f^*, $\mathcal{A}^{\#}$, as the connected component of the basin containing z_f^*. It is well-known if z_f^* is an superattracting fixed point, the immediate basin of attraction $\mathcal{A}^{\#}$ contains at least a critical point.

In order to study the affine conjugacy classes of the iterative methods, the following relevant result must be mentioned.

Theorem 1 (Scaling Theorem for Newton's method, [2]). *Let $g(z)$ be an analytic function, and let $A(z) = \alpha z + \beta$, with $\alpha \neq 0$, be an affine map. Let $h(z) = \lambda(g \circ A)(z)$, with $\lambda \neq 0$. Let $O_p(z)$ be the fixed point operator of Newton's method. Then, $A \circ O_h \circ A^{-1}(z) = O_g(z)$, i.e., O_g and O_h affine conjugated by A.*