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Diagonal imbeddings in a normal matrix

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ABSTRACT

Let a normal matrix $A \in \mathbb{C}^{n\times n}$ and μ_1 a given point in its numerical range $w(A)$ that is not an eigenvalue. In this paper, we study the problem of construction of an isometry $W \in \mathbb{C}^{n\times(n-k)}$ $(1 \leqslant k < n)$, such that $W^*AW = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_{n-k})$, where the μ_j's do not belong to the spectrum $\sigma(A)$, for $j = 2, \ldots, n-k$. In particular, the smallest integer k is determined, such that an isometry W exists, and our approach for W is based on the construction of $(n-k)$ mutually orthogonal and A-orthogonal vectors of $\mathbb{C}^n$.

1. Introduction

The *numerical range* of a matrix $A \in \mathbb{C}^{n\times n}$ is a closed, convex subset of the complex plane defined by

$$w(A) = \{x^*Ax : x \in \mathbb{C}^n, \|x\| = 1\}$$

and when $A \in \mathbb{C}^{n\times n}$ is normal, $w(A)$ coincides with the convex hull of its eigenvalues, i.e. $w(A) = co\{\sigma(A)\}$. This set has attracted a lot of interest, especially in the context of numerical solutions of partial differential equations, stability analysis of dynamical systems and convergence theory of matrix iterations, among other applications. For an introduction to this area, see [7,8].

Given $\mu \in w(A)$, a unit vector $x \in \mathbb{C}^n$ is a *generating vector* for μ if $x^*Ax = \mu$. Recently in [15], for a given matrix A and a point $\mu \in w(A)$ the inverse problem of finding a generating vector x for μ was studied. This is a problem relating the image $w(A)$ of the quadratic map

$$x \in \mathbb{C}^n,\ \|x\| = 1 \ \rightarrow\ x^*Ax \in \mathbb{C}$$

to its domain, the complex unit sphere $\mathcal{S} = \{x \in \mathbb{C}^n : \|x\| = 1\}$ and it has been proved in [2] that there exist n linearly independent generating vectors for any such point. Algorithms computing generating vectors can be found in [15,2].

For the pair of matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{(n-k)\times(n-k)}$ $(1 \leqslant k < n)$, we say that B is *imbeddable* in A if there exists an isometry $W \in \mathbb{C}^{n\times(n-k)}$ (i.e. $W^*W = I_{n-k}$) such that $W^*AW = B$. The problem of imbedding for Hermitian matrices has been investigated in [6], while several necessary imbedding conditions for a normal matrix B to be imbeddable in normal A and related results have been presented in [5,14,9–11]. In the special case $B = \lambda I_{n-k}$, this problem has been studied via *higher rank numerical ranges* [3],

$$\Lambda_{n-k}(A) = \{\lambda \in \mathbb{C} : PAP = \lambda P,\ \text{for some rank} - (n-k)\ \textit{projection}\ P\} = \{\lambda \in \mathbb{C} : \lambda I_{n-k}\ \text{is imbeddable in}\ A\},$$

which have recently found applications in *quantum error correction.*

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In this paper, we propose a new and more general inverse numerical range problem, involving the construction of subspaces of dimension greater than one that yield diagonal matrices imbeddable in A. For a normal matrix $A \in \mathbb{C}^{n\times n}$ and a point $\mu_1 \in w(A) \setminus \sigma(A)$, we seek to determine the minimum integer $1 \leqslant k < n$, such that the relationship $W^*AW = \operatorname{diag}(\mu_1, \mu_2, \dots, \mu_{n-k})$ is satisfied, for an isometry $W \in \mathbb{C}^{n\times(n-k)}$ and suitable, not unique points $\mu_2, \dots, \mu_{n-k} \in w(A) \setminus \sigma(A)$ to be determined. In particular, for the smallest k we construct $n-k$ mutually orthonormal and A-orthogonal vectors $w_j \in \mathbb{C}^n$ (i.e. $w_i^* w_j = w_i^* A w_j = w_i^* A^* w_j = 0$, for $i \neq j$) that define the isometry $W = [\,w_1 \quad w_2 \quad \cdots \quad w_{n-k}\,] \in \mathbb{C}^{n\times(n-k)}$. Then, $B = \operatorname{diag}\left\{\mu_j\right\}_{j=1}^{n-k}$ is a matrix of maximum order imbeddable in A and W will be referred to as *generating isometry* for the diagonal B in the normal A. It should be pointed out that nothing relevant to this problem has been presented in the literature. Our results may be helpful for the construction of generating isometries corresponding to points in higher rank numerical ranges. This is crucial for the construction of error correcting codes in quantum computing and remains an open problem in this area [4].

Our paper is organized as follows: in Section 2 the smallest integer k for which a diagonal matrix should be imbeddable in A is determined and a counterexample shows that this bound is the best possible. Further in Section 3, a recursive procedure to produce $(n-k)$ mutually orthogonal and A-orthogonal unit vectors is proposed and consequently an isometry $W \in \mathbb{C}^{n\times(n-k)}$ is constructed, such that the matrix W^*AW is diagonal. Moreover, some useful properties, derived from diagonal imbeddability, are presented. In the last section, we conclude our paper presenting a generalization of a result in [1], which implies a method to obtain generating vectors for points $\mu_1 \in \operatorname{int}\{w(A)\}$ and to construct chains of orthonormal bases for subspaces in which the columns of W recursively lie.

2. Maximality of a diagonal matrix imbedded in a normal matrix

Let a normal matrix $A \in \mathbb{C}^{n\times n}$ and a point $\mu_1 \in w(A)$ that is not an eigenvalue. We consider in this section the problem to find the minimum integer $1 \leqslant k < n$ and an isometry $W \in \mathbb{C}^{n\times(n-k)}$ with the property that $W^*AW = \operatorname{diag}(\mu_1, \mu_2, \dots, \mu_{n-k})$ for suitable points $\mu_2, \dots, \mu_{n-k} \in w(A) \setminus \sigma(A)$.

In the case the eigenvalues $\{\lambda_j\}_{j=1}^n$ of A are collinear, the polygon $w(A)$ degenerates to a line segment. If $\mathcal{L}$ denotes the line on which the eigenvalues of A lie, $\phi \in [0, 2\pi)$ is the slope of $\mathcal{L}$ and $z_0 \in \mathbb{R} \cap \mathcal{L}$, then the matrix A is equal to $z_0\, I_n + e^{\mathbf{i}\phi}\, H$, where $H \in \mathbb{C}^{n\times n}$ is Hermitian. Hence by a translation, followed by a rotation, there is no loss of generality in assuming that A is Hermitian with real eigenvalues $\{\lambda_j\}_{j=1}^n$. In this case the minimum value of k is $k = 1$ and the λ's and μ's are interlacing; see [6,11]. In the following we focus on normal matrices A, whose numerical range is a non-degenerate polygon and may restrict our attention to the case $n \geqslant 4$, since it is well known by Theorem 2 in [6] that a matrix $B = \operatorname{diag}(\mu_1, \mu_2)$ with $\mu_1, \mu_2 \notin \sigma(A)$ is imbeddable in a normal $A \in \mathbb{C}^{3\times 3}$ if and only if its eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and μ_1, μ_2 are collinear and interlacing.

The next Theorem describes certain restrictions for the existence of mutually orthogonal and A-orthogonal unit vectors $\{w_\ell\}_{\ell=1}^{j+1}$, when $j+1 \leqslant n-k$ and then we are led to the determination of the minimum value of k. Notice that our first result is true in general and the matrix A need not be normal.

Theorem 1. *Let the matrix $A \in \mathbb{C}^{n\times n}$ with $n \geqslant 4$ and a positive integer $j \in [1, \frac{n-1}{3}]$ such that for the isometry $W_j = [\,w_1 \quad \cdots \quad w_j\,] \in \mathbb{C}^{n\times j}$ the matrices A and $\operatorname{diag}\left(\mu_1, \dots, \mu_j\right)$ are imbeddable, where $\mu_1, \dots, \mu_j \in w(A) \setminus \sigma(A)$. Then there exists a point $\mu_{j+1} = w_{j+1}^* A w_{j+1}$ such that*

$$\operatorname{diag}\left(\mu_1, \dots, \mu_j, \mu_{j+1}\right) = W_{j+1}^* A W_{j+1},$$

where $W_{j+1} = [\,W_j \quad w_{j+1}\,] \in \mathbb{C}^{n\times(j+1)}$ is an isometry.

Proof. Notice that the subspaces $\mathcal{S}_{j,1} = \operatorname{Ker}(W_j^*), \mathcal{S}_{j,2} = \operatorname{Ker}(W_j^* A)$ and $\mathcal{S}_{j,3} = \operatorname{Ker}(W_j^* A^*)$ have nontrivial intersection, due to the dimensional inequality

$$\dim\left(\mathcal{S}_{j,1} \cap \mathcal{S}_{j,2} \cap \mathcal{S}_{j,3}\right) = \sum_{i=1}^{3} \dim \mathcal{S}_{j,i} - \dim\left(\mathcal{S}_{j,2} + \mathcal{S}_{j,3}\right) - \dim\left(\left(\mathcal{S}_{j,2} \cap \mathcal{S}_{j,3}\right) + \mathcal{S}_{j,1}\right) \geqslant 3(n-j) - 2n = n - 3j \geqslant 1.$$

Denoting by $\mathcal{T}_j \equiv \mathcal{S}_{j,1} \cap \mathcal{S}_{j,2} \cap \mathcal{S}_{j,3}$ and $w_{j+1} \in \mathcal{T}_j$ the generating vector for $\mu_{j+1} = w_{j+1}^* A w_{j+1}$, then clearly the matrix $W_{j+1} = [\,W_j \quad w_{j+1}\,]$ is an isometry, since $w_{j+1} \in \mathcal{S}_{j,1}$. Moreover, we have

$$W_{j+1}^* A W_{j+1} = \begin{bmatrix} W_j^* A W_j & W_j^* A w_{j+1} \\ w_{j+1}^* A W_j & w_{j+1}^* A w_{j+1} \end{bmatrix} = \begin{bmatrix} \operatorname{diag}\left(\mu_1, \dots, \mu_j\right) & 0 \\ 0 & \mu_{j+1} \end{bmatrix}$$

as desired, due to $w_{j+1} \in \mathcal{S}_{j,2} \cap \mathcal{S}_{j,3}$. $\square$

For $n = 3$, the following result gives necessary and sufficient conditions for the existence of a 2×2 diagonal matrix imbeddable in an arbitrary matrix $A \in \mathbb{C}^{3\times 3}$. Given a vector w_1, we denote, as in Theorem 1, the subspaces