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Solving Fredholm integro–differential equations using reproducing kernel Hilbert space method

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ABSTRACT

In this study, the numerical solution of Fredholm integro–differential equation is discussed in a reproducing kernel Hilbert space. A reproducing kernel Hilbert space is constructed, in which the initial condition of the problem is satisfied. The exact solution $u(x)$ is represented in the form of series in the space $W_2^2[a,b]$. In the mean time, the n-term approximate solution $u_n(x)$ is obtained and is proved to converge to the exact solution $u(x)$. Furthermore, we present an iterative method for obtaining the solution in the space $W_2^2[a,b]$. Some examples are displayed to demonstrate the validity and applicability of the proposed method. The numerical result indicates that the proposed method is straightforward to implement, efficient, and accurate for solving linear and nonlinear Fredholm integro–differential equations.

1. Introduction

Integro–differential equation (IDE) has a great deal of application in different branches of sciences and engineering. It arises naturally in a variety of models from biological science, applied mathematics, physics, and other disciplines, such as theory of elasticity, biomechanics, electromagnetic, electrodynamics, fluid dynamics, heat and mass transfer, oscillating magnetic field, etc. [1–4]. This class of equations is sometimes too complicated to be solved exactly because, generally, the solution cannot be exhibited in a closed form even when it exists. Therefore, finding either the analytical approximation or numerical solution of such equations are of great interest.

In this paper, we are concerned with providing the numerical solution based on the use of reproducing kernel Hilbert space (RKHS) method for Fredholm IDEs of the general form

$$\frac{d}{dx}u(x) = F(x,u(x)) + Tu(x), \quad a \leqslant x, \quad t \leqslant b, \tag{1}$$

where

$$Tu(x) = \int_a^b K(x,t)G(u(t))dt,$$

subject to the initial condition

$$u(a) = \alpha, \tag{2}$$

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where $a, b, \alpha \in \mathbb{R}, u \in W_2^2[a,b]$ is an unknown function to be determined, $K(x,t)$ is continuous function on $[a,b] \times [a,b], F(x,y), G(z)$ are continuous terms in $W_2^1[a,b]$ as $y = y(x), z = z(x) \in W_2^2[a,b], a \leqslant x \leqslant b, -\infty < y, z < \infty$ and are depending on the problem discussed, and $W_2^1[a,b], W_2^2[a,b]$ are two reproducing kernel spaces.

The numerical solvability of Fredholm IDEs and other related equations has been pursued by several authors. To mention a few, in [5] the authors have discussed the Cattani's method for solving Fredholm IDE $u'(x) = u(x) + f(x) + \int_a^b K(x,t)u(t)dt$. In [6] also, the authors have provided the Tau method to further investigation to IDE $u'(x) = f(x) + \int_a^b K(x,t)G(u(t))dt$. Furthermore, the homotopy analysis, differential transformation, and Sinc functions methods are carried out in [7–9] for the Fredholm IDE $u'(x) = u(x)g(x) + f(x) + \int_a^b K(x,t)u(t)dt$. The homotopy perturbation method has been applied to solve Fredholm equation $u'(x) = f(x) + \int_a^b K(x,t)G(u(t), u'(t))dt$ as described in [10]. Recently, the sequential approach method for solving equation $u'(x) = f(x) + \int_a^b K(x,t)G(u(t))dt$ is proposed in [11]. However, none of previous studies propose a methodical way to solve these equations. Moreover, previous studies require more effort to achieve the results, they are not accurate and usually they are developed for special types of Eqs. (1) and (2). On the other hand, the proposed method has an advantage that it is possible to pick any point in the interval of integration and as well the approximate solution and its derivative will be applicable.

The theory of reproducing kernel has recently emerged as a powerful framework in numerical analysis, differential and integral equations, and probability and statistics [12–14]. On the other aspects as well, a RKHS is a useful framework for constructing approximate solutions for linear and nonlinear equations. This method has been implemented in several differential, integral, integro–differential, operator, and system of equations, such as singular boundary value problems [15–17], system of boundary value problems [18], partial differential equations [19,20], Volterra–Fredholm integral equations [21], singular integral equations [22], Fredholm–Volterra IDEs [23,24], operator equations [25], infinite system of equations [26,27], and others.

This paper is organized in six sections including the introduction. In Section 2, two reproducing kernel spaces are presented in order to construct a reproducing kernel function in the space $W_2^2[a,b]$. In Section 3, the analytical solution for Eqs. (1) and (2) in the space $W_2^2[a,b]$ and some essential results are introduced. Also, an iterative method to solve Eqs. (1) and (2) numerically in the space $W_2^2[a,b]$ is described. In Section 4, the n-term approximate solution $u_n(x)$ is proved to converge to the exact solution $u(x)$ in the space $W_2^2[a,b]$. Numerical experiments are presented in Section 5. Finally, in Section 6 some concluding remarks are presented.

2. Construction of reproducing kernel function

In this section, we construct a reproducing kernel function in order to solve Eqs. (1) and (2) using RKHS method in the space $W_2^2[a,b]$. First of all, an abstract set is supposed to have elements, each of which has no structure, and is itself supposed to have no internal structure, except that the elements can be distinguished as equal or unequal, and to have no external structure except for the number of elements.

Definition 1 [15]. Let E be a nonempty abstract set and $\mathbb{C}$ be the set of complex numbers. A function $K : E \times E \to \mathbb{C}$ is a reproducing kernel of the Hilbert space H if

1. $K(\cdot, t) \in H$ for all $t \in E$,
2. $\langle \varphi(\cdot), K(\cdot, t)\rangle = \varphi(t)$ for all $t \in E$ and all $\varphi \in H$.

The name reproducing kernel is inspired by the reproducing property (2) above, which means that the value of the function φ at the point t is reproducing by the inner product of φ with $K(\cdot, t)$. A Hilbert space which possesses a reproducing kernel is called a RKHS [15].

Next, we first construct the space $W_2^2[a,b]$ in which every function satisfies the initial condition (2) and then formulate the reproducing kernel function $K_x(y)$ in the space $W_2^2[a,b]$. Here, $L^2[a,b] = \{u \mid \int_a^b u^2(x)dx < \infty\}$.

Definition 2 [20]. $W_2^2[a,b] = \{u : u, u'$ is absolutely continuous on $[a,b], u, u', u'' \in L^2[a,b]$, and $u(a) = 0\}$. The inner product and the norm in $W_2^2[a,b]$ are defined respectively by

$$\langle u, v\rangle_{W_2^2} = u(a)v(a) + u'(a)v'(a) + \int_a^b u''(y)v''(y)dy \tag{3}$$

and $\|u\|_{W_2^2} = \sqrt{\langle u, u\rangle_{W_2^2}}$, where $u, v \in W_2^2[a,b]$.

It easy to see that $\langle u, v\rangle_{W_2^2}$ satisfies all the requirements for the inner product. First, $\langle u, u\rangle_{W_2^2} \geqslant 0$. Second, $\langle u, v\rangle_{W_2^2} = \langle v, u\rangle_{W_2^2}$. Third, $\langle \gamma u, v\rangle_{W_2^2} = \gamma\langle u, v\rangle_{W_2^2}$. Fourth, $\langle u + w, v\rangle_{W_2^2} = \langle u, v\rangle_{W_2^2} + \langle w, v\rangle_{W_2^2}$, where $u, v, w \in W_2^2[a,b]$. It therefore remains only to prove that $\langle u, u\rangle_{W_2^2} = 0$ if and only if $u = 0$. In fact, it is obvious that when $u = 0$ then $\langle u, u\rangle_{W_2^2} = 0$. On the other hand, if $\langle u, u\rangle_{W_2^2} = 0$, then by Eq. (3), we have $\langle u, u\rangle_{W_2^2} = (u(a))^2 + (u'(a))^2 + \int_a^b (u''(y))^2 dy = 0$, therefore, $u(a) = u'(a) = 0$ and $u''(y) = 0$. Then, we can obtain $u(y) = 0$.