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A posteriori error estimates of stabilized finite element method for the steady Navier–Stokes problem ☆

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ABSTRACT

In this paper we develop a posteriori error estimates for the steady Navier–Stokes equations based on the lowest equal-order mixed finite element pair. Residual type a posteriori error estimates are derived by means of general framework established by Verfürth for the nonlinear equations. Furthermore, a simple error estimator in L^2 norm is also presented by using the duality argument. Numerical experiments using adaptive computations are presented to demonstrate the effectiveness of these error estimates for three examples. The first example is a singular problem with known solution, the second example is a physical model of lid driven cavity and the last one is a backward facing step problem.

1. Introduction

In the numerical approximation of partial differential equations, one usually faces the problem of increasing the accuracy of a solution without adding unnecessary degrees of freedom, therefore, we need to update the mesh to ensure that the mesh becomes fine enough in the critical regions while remaining reasonable coarse in the rest of the domain. In recent years, a lot of work have been done about the adaptive mesh refinement techniques based on a posteriori error estimators for finite element discretizations of 2nd order partial differential equations, see [1,4,7,8,11] and the reference therein.

Deriving an efficient posteriori indicators for the Stokes equations have attracted much attentions. The works of Verfürth [25] provided a basic foundation for the mathematical theory of practical methods, and this problem has also been advanced by Ainsworth and Oden [2], He et al. [18], Kondratyuk and Stevenson [21] and others. For the nonlinear problems, Verfürth has established a general framework in [26] and proved that they were global upper bound and local lower bound for the finite element errors. These indicators can be changed to be applied to other situations, see [3,15,23]. For the Navier–Stokes equations, the importance of ensuring the compatibility of the approximations for velocity and pressure by satisfying the so-called inf–sup condition is widely understood [17,24]. Although some stable mixed finite element pairs have been studied over the years, the low order mixed finite element pairs not satisfying the inf–sup condition may work well. For example, those pairs have simple constructions and high efficiency of computation. In order to use these finite element pairs, various stabilized techniques have been proposed and studied. For instance, the pressure gradient method [5], the polynomial pressure projection method [6,12,28], the Douglas–Wang method [13], the method of local Gauss integrations [22] and so

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on. Based on the attractive features of the adaptive method and stabilized method, many researchers have combined them to treat the linear and nonlinear problems, see references [14,20,29] and therein.

In this paper, under the framework of Verfürth [26], we describe two residual types of error estimators and derive a posteriori error estimates for the steady Navier–Stokes equations based on the P_1-P_1 element. The reasons of choosing P_1-P_1 element lie in: (1) The P_1-P_1 pair is computationally convenient in a parallel processing and multigrid context because this pair holds the identical distribution for both the velocity and pressure; (2) The P_1-P_1 pair is of practical importance in scientific computation with the lowest computational cost. We obtain the global upper and local lower bounds for the velocity and pressure in energy norm. Furthermore, with the help of the duality argument, a global upper bound for the velocity in L^2-norm is also derived.

This paper is organized as follows. Section 2 is devoted to present some notations and preliminary results for the steady Navier–Stokes equations. In Section 3, an abstract framework from the literature [26] for constructing a posteriori error estimates for nonlinear equations is presented. After describing some finite element tools necessary for constructing the error estimates, the stabilized finite element formulation for the Navier–Stokes equations is then cast in this framework. A posteriori error estimates for numerical solution in energy norm and L^2-norm are constructed in Section 4. In Section 5, some numerical results are given to illustrate the efficiency of the established error estimators. Finally, some conclusions are given in Section 6.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^2$ assumed to have a Lipschitz continuous boundary $\partial\Omega$. We consider the following incompressible flow problem:

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } \Omega,\\ \text{div } u = 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $u = (u_1(x), u_2(x))^T$ represents the velocity, $p = p(x)$ the pressure, $f = f(x)$ the prescribed body force and $\nu > 0$ the viscosity.

In order to simple the notations, we set

$$X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \right\}.$$

Here, $W^{l,q}(\Omega)$ be a standard Sobolev spaces with norm and seminorm $\|\cdot\|_{l,q,\Omega}$ and $|\cdot|_{l,q,\Omega}$, respectively. The spaces $L^2(\Omega)^m$ $(m = 1,2)$ are endowed with the standard L^2-scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|_{0,\Omega}$. The spaces $H_0^1(\Omega)$ and X are equipped with the scalar product $(\nabla u, \nabla v)$ and norm $\|u\|_{1,\Omega}^2 = (\nabla u, \nabla u), \forall\, u, v \in H_0^1(\Omega)$ or X. Due to the norms equivalence between $\|u\|_{1,\Omega}$ and $|u|_{1,\Omega}$ on $H_0^1(\Omega)^i (i = 1,2)$, we use the same notation for them.

We define the generalized bilinear form on $(X,M) \times (X,M)$ by

$$B((u,p);(v,q)) = a(u,v) - d(v,p) + d(u,q) = \nu(\nabla u, \nabla v) - (p, \text{div} v) + (q, \text{div} u)$$

and the trilinear form on $X \times X \times X$

$$b(u,v,w) = ((u\cdot\nabla)v, w) + \frac{1}{2}((\text{div} u)v, w) = \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v), \quad \forall\, u, v, w \in X.$$

If $\partial\Omega$ is of C^2 or Ω is a two-dimensional convex polygon, the following inequalities which are borrowed from [10] hold:

$$\|v\|_0 \leqslant \gamma_0 |v|_1, \quad \|v\|_{0,4} \leqslant 2^{1/4}\|v\|_{0,2}^{1/2}\|\nabla v\|_{0,2}^{1/2}, \quad \forall\, v \in X; \tag{2.2}$$

where γ_0 is a positive constant only depending on Ω.

It is easy to verify that $B((\cdot,\cdot);(\cdot,\cdot))$ and $b(\cdot,\cdot,\cdot)$ satisfy the following important properties (see [17,19,24]):

$$\begin{cases} B((u,p);(u,p)) = \nu|u|_1^2,\\ |B((u,p);(v,q))| \leqslant C(|u|_1 + \|p\|_0)(|v|_1 + \|q\|_0),\\ \beta_0(|u|_1 + \|p\|_0) \leqslant \sup_{(v,q)\in(X,M)} \frac{|B((u,p);(v,q))|}{|v|_1 + \|q\|_0} \end{cases} \tag{2.3}$$

for all $(u,p),(v,q) \in (X,M)$, and the constant $\beta_0 > 0$,

$$b(u,v,w) = -b(u,w,v), \quad b(u,v,v) = 0, \tag{2.4}$$

$$|b(u,v,w)| \leqslant N\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0 \tag{2.5}$$

for all $u, v, w \in X$, where

$$N = \sup_{u,v,w\in X} \frac{|b(u,v,w)|}{\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0}.$$