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On a system of difference equations which can be solved in closed form

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ABSTRACT

We present an elegant method for solving the system of difference equations

$$x_{n+1}=\frac{1}{x_n}+\frac{\alpha}{y_n},\quad y_{n+1}=\frac{\beta}{x_n}+\frac{1}{y_n},\quad n\in\mathbb{N}_0,$$

where parameters α and β and the initial values x_0 and y_0 are positive numbers, and by using obtained formulae we show that every solution of the system converges to a two-periodic solution.

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1. Introduction

Recently there has been a considerable interest in studying systems of difference equations (see, for example, [6,11–16,18,28,31–34,36,37] and the related references therein). Beside this there has been some renewed interest in equations and systems which can be solved in closed form and in their applications (see, for example, [1,2,7,17,23,24,26,28,29,31,32,34–37] and the related references therein). For some classical methods for solving difference equations and systems see, for example, book [10].

The following conjecture was posed in 2000 by G. Ladas.

Conjecture 1. *Consider the following system of difference equations*

$$x_{n+1}=\frac{1}{x_n}+\frac{\alpha}{y_n},\quad y_{n+1}=\frac{\beta}{x_n}+\frac{1}{y_n},\quad n\in\mathbb{N}_0,\tag{1}$$

where $x_0, y_0, \alpha, \beta > 0$. *Then every positive solution of system* (1) *converges to a two-periodic solution of the system.*

Two different solutions of the conjecture were independently presented in the same year ([5,19]). Here we present our original proof of the conjecture, which although has fluctuated among some experts since 2000, has not been published so far. Above mentioned renewed interest in equations and systems which can be solved in closed form, recollected us to our original proof of the conjecture, which along with some nice reactions of several experts to our solution of the conjecture, motivated us to present the solution to wide audience. Namely, we confirm the conjecture by finding solution $(x_n, y_n)_{n\in\mathbb{N}_0}$ of system (1) in closed form, in a direct and elegant way, without using some other difference equations as it is the case in [5].

We say that a solution $(x_n, y_n)_{n\in\mathbb{N}_0}$, of system (1) is eventually periodic with period p, if there is an $n_1 \geqslant 0$ such that

$$x_{n+p}=x_n \text{ and } y_{n+p}=y_n \quad \text{for } n\geqslant n_1.$$

If $n_1 = 0$, then for the solution is said that it is periodic with period p. For some periodic difference equations and systems, as well as some results on periodicity, see, for example, [3,4,8,9,12–14,20–22,25,27,30,33] and the related references therein.

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Remark 1. Note that the following system of difference equations

$$\bar{x}_{n+1} = \frac{a}{\bar{x}_n} + \frac{b}{\bar{y}_n}, \quad \bar{y}_{n+1} = \frac{c}{\bar{x}_n} + \frac{d}{\bar{y}_n}, \quad n \in \mathbb{N}_0, \tag{2}$$

where parameters a, b, c, d and initial values $\bar{x}_0$ and $\bar{y}_0$ are positive, is transformed into a system of form (1) by using the change of variables

$$\bar{x}_n = \sqrt{a} x_n, \quad \bar{y}_n = \sqrt{d} y_n.$$

For this and since system (1) is written in a bit more compact form, instead of system (2) we consider system (1).

2. Main results

Here we formulate and prove our main results.

Lemma 1. *Assume that $x_0, y_0, \alpha, \beta > 0$. Then, system of difference equations (1) can be solved in closed form.*

Proof. First, it is easy to show by induction that $x_n, y_n > 0$, for all $n \in \mathbb{N}_0$. Multiplying the second equation in (1) by $\theta \neq 0$, then adding such obtained equation to the first one, we obtain

$$x_{n+1} + \theta y_{n+1} = \frac{(\alpha + \theta)x_n + (1 + \beta\theta)y_n}{x_n y_n}, \quad n \in \mathbb{N}_0. \tag{3}$$

We choose $\theta \neq 0$ such that

$$1 + \beta\theta = \theta(\alpha + \theta). \tag{4}$$

Hence

$$\theta_1 = \frac{\beta - \alpha + \sqrt{(\beta - \alpha)^2 + 4}}{2} \text{ and } \theta_2 = \frac{\beta - \alpha - \sqrt{(\beta - \alpha)^2 + 4}}{2},$$

from which it is easy to see that $\theta_1 \neq 0 \neq \theta_2$.

To use the relations in (3) it is of interest that $\alpha + \theta_i \neq 0, i = 1, 2$, which is equivalent to

$$\alpha + \beta \pm \sqrt{(\beta - \alpha)^2 + 4} \neq 0,$$

which is the case when $\alpha\beta \neq 1$. Note that if $\beta = 1/\alpha$, then $\theta_1 = 1/\alpha$ and $\theta_2 = -\alpha$.

Hence, first we assume that $\alpha\beta \neq 1$. From (3) and (4) we obtain

$$x_{n+1} + \theta_i y_{n+1} = (\alpha + \theta_i)\frac{x_n + \theta_i y_n}{x_n y_n}, \quad n \in \mathbb{N}_0, \quad i = 1, 2, \tag{5}$$

and consequently

$$\frac{x_{n+1} + \theta_1 y_{n+1}}{x_{n+1} + \theta_2 y_{n+1}} = \frac{\alpha + \theta_1}{\alpha + \theta_2} \cdot \frac{x_n + \theta_1 y_n}{x_n + \theta_2 y_n}, \quad n \in \mathbb{N}_0. \tag{6}$$

Set

$$u_n = \frac{x_n + \theta_1 y_n}{x_n + \theta_2 y_n}, \quad n \in \mathbb{N}_0. \tag{7}$$

Then from (6) we obtain

$$u_n = \left(\frac{\alpha + \theta_1}{\alpha + \theta_2}\right)^n u_0, \tag{8}$$

if $x_0 \neq -\theta_2 y_0$.

Let

$$c := \frac{\alpha + \theta_1}{\alpha + \theta_2}.$$

Then it is easy to see that

$$|c| > 1. \tag{9}$$

From (7) and (8) we obtain

$$x_n(1 - c^n u_0) = y_n(\theta_2 c^n u_0 - \theta_1), \quad n \in \mathbb{N}_0. \tag{10}$$

Substituting (10) in (1) we obtain that for every $n \in \mathbb{N}_0$

$$x_{n+1} = \frac{1}{x_n}\left(1 + \alpha\frac{\theta_2 c^n \left(\frac{x_0 + \theta_1 y_0}{x_0 + \theta_2 y_0}\right) - \theta_1}{1 - c^n\left(\frac{x_0 + \theta_1 y_0}{x_0 + \theta_2 y_0}\right)}\right) = \frac{1 - \alpha\theta_2}{x_n}\left(1 + \frac{\alpha(\theta_1 - \theta_2)}{1 - \alpha\theta_2}\,\frac{1}{c^n\left(\frac{x_0 + \theta_1 y_0}{x_0 + \theta_2 y_0}\right) - 1}\right). \tag{11}$$

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