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Global exponential stability of nonautonomous neural network models with continuous distributed delays

Salete Esteves [a], Elçin Gökmen [b], José J. Oliveira [c,*]

[a] Departamento de Informática e Matemática, EsACT – IPB, 5370-326 Mirandela, Portugal
[b] Department of Mathematics, Faculty of Science, Muğla Sıtkı Koçman University, 4800 Muğla, Turkey
[c] Departamento de Matemática e Aplicações and CMAT, Escola de Ciências, Universidade do Minho, Campus de Gualtar, 4710-057 Braga, Portugal

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ABSTRACT

For a family of non-autonomous differential equations with distributed delays, we give sufficient conditions for the global exponential stability of an equilibrium point. This family includes most of the delayed models of neural networks of Hopfield type, with time-varying coefficients and distributed delays. For these models, we establish sufficient conditions for their global exponential stability. The existence and global exponential stability of a periodic solution is also addressed. A comparison of results shows that these results are general, news, and add something new to some earlier publications.

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1. Introduction

In the last decades, retarded functional differential equations (FDEs) have attracted the attention of an increasing number of scientists due to their potential applications in different sciences. Differential equations with delays have served as models in population dynamics, ecology, epidemiology, disease modeling, and neural networks.

Neural network models possess good potential applications in areas such as content-addressable memory, pattern recognition, signal and image processing and optimization (see [2,14,17,18], and references therein).

In 1983, Cohen and Grossberg [5] proposed and studied the artificial neural network described by a system of ordinary differential equations

$$x_i'(t) = -k_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij}f_j(x_j(t))\right], \quad i = 1, \ldots, n \tag{1.1}$$

and, in 1984, Hopfield [9] studied the particular situation of (1.1) with $k_i \equiv 1$,

$$x_i'(t) = -b_i x_i(t) + \sum_{j=1}^{n} a_{ij}f_j(x_j(t)), \quad i = 1, \ldots, n. \tag{1.2}$$

In 1988, Kosko presented a kind of neural networks, which is called bidirectional associative memory (BAM) neural network, [12],

* Corresponding author.
E-mail addresses: saleteesteves@ipb.pt (S. Esteves), elcingokmen@hotmail.com (E. Gökmen), jjoliveira@math.uminho.pt (J.J. Oliveira).

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$$\begin{cases} x_i'(t) = -x_i(t) + \sum_{j=1}^{n} a_{ij} f(y_j(t)) + I_i \\ y_i'(t) = -y_i(t) + \sum_{j=1}^{n} b_{ji} f(x_i(t)) + J_i \end{cases}, \quad i = 1, \ldots, n. \tag{1.3}$$

The finite switching speed of the amplifiers, communication time, and process of moving images led to the use of time-delays in models (1.1), (1.2), and (1.3), arising the delayed neural network models. In the applications of delayed neural networks to some practical problems, stability plays an important role. It is well known that delays can affect the dynamic behavior of neural networks (see [1,13]). For this reason, stability of delayed neural networks has been investigated extensively. There are many important results on static (equilibrium-type) attractors of neural networks (see [2,3,7,10,14,15], and the references therein), but it is well known that non-static attractors, such as periodic oscillatory behavior, are also an important aspect (see [4,11,16,19,20], and the references therein).

In the literature, the usual approach to analyze the stability property is to construct a suitable Lyapunov functional for a concrete n-dimensional FDE and then to derive sufficient conditions ensuring stability. However, constructing a Lyapunov functional is not an easy task and, frequently, a new functional is required for each model under consideration. In quite an unusual way, our techniques (see [6,7,14,15]) do not involve Lyapunov functionals, and our method applies to general systems.

This paper is organized as follows: In Section 2, we briefly present the phase space for FDEs written in abstract form as $x'(t) = f(t, x_t)$, then we define the global exponential stability of a FDE, and finally we establish a general condition for the boundedness of solutions of $x'(t) = f(t, x_t)$. In Section 3, we present the results on global exponential stability of a general class of nonautonomous delay differential equations, which includes most of neural network models. In Section 4, we prove the existence and global exponential stability of a periodic solution of a periodic general Hopfield neural network type model. Finally, in Section 5, we illustrate the results with well-known nonautonomous n-dimensional neural network models and we compare our results with the literature, showing the advantage of our method when applied to several different models, such as Hopfield or BAM neural network models.

2. Preliminaries

For $a, b \in \mathbb{R}$ with $b > a$ and $n \in \mathbb{N}$, we denote by $C([a, b]; \mathbb{R}^n)$ the vector space of continuous functions $\varphi : [a, b] \to \mathbb{R}^n$, equipped with the supremum norm $\|\cdot\|$ relative to the max norm $|\cdot|$ in $\mathbb{R}^n$, i.e., $\|\varphi\| = \sup_{a \leqslant \theta \leqslant b} |\varphi(\theta)|$ for $\varphi \in C([a, b]; \mathbb{R}^n)$, where $|x| = \max_{i=1,\ldots,n} |x_i|$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For $c \in \mathbb{R}$, we use c to denote the constant function $\varphi(\theta) = c$ in $C([a, b]; \mathbb{R}^n)$. A vector $d = (d_1, \ldots, d_n) \in \mathbb{R}^n$ is said to be positive if $d_i > 0$ for $i = 1, \ldots, n$, and in this case we write $d > 0$.

In the space $C_n := C([-\tau, 0]; \mathbb{R}^n)$, for $\tau > 0$, consider FDEs,

$$x'(t) = f(t, x_t), \quad t \geqslant 0, \tag{2.1}$$

where $f : [0, +\infty) \times C_n \to \mathbb{R}^n$ is a continuous function and, as usual, x_t denotes the function in C_n defined by $x_t(\theta) = x(t + \theta), \ -\tau \leqslant \theta \leqslant 0$. It is well-known that, assuming the Banach space C_n as the phase space of (2.1), the standard existence, uniqueness, and continuous type results are valid (see [8]). We always assume that f is regular enough in order to have uniqueness of solutions for the initial value problem. The solution of (2.1) with initial condition $x_{t_0} = \varphi$, for $t_0 \geqslant 0$ and $\varphi \in C_n$, is denoted by $x(t, t_0, \varphi)$. For $\omega > 0$ and $\varphi \in C_n$, we write $x_\omega(\varphi)$, or just x_ω if there is no confusion, to denote the function in C_n defined by $x_\omega(\varphi)(\theta) = x(\omega + \theta, 0, \varphi), \ \theta \in [-\tau, 0]$.

Definition 2.1. The solution $x(t, 0, \bar{\varphi})$ of (2.1), with $\bar{\varphi} \in C_n$, is said globally exponentially stable if there are $\varepsilon > 0$ and $M \geqslant 1$ such that

$$|x(t, 0, \varphi) - x(t, 0, \bar{\varphi})| \leqslant M e^{-\varepsilon t} \|\varphi - \bar{\varphi}\|, \quad \forall t \geqslant 0, \ \forall \varphi \in C_n.$$

Definition 2.2. The system (2.1) is said globally exponentially stable if there are $\varepsilon > 0$ and $M \geqslant 1$ such that

$$|x(t, 0, \varphi_1) - x(t, 0, \varphi_2)| \leqslant M e^{-\varepsilon t} \|\varphi_1 - \varphi_2\|, \quad \forall t \geqslant 0, \ \forall \varphi_1, \varphi_2 \in C_n.$$

In [6], a relevant result on the boundedness of solutions for the general FDE (2.1) was established. For convenience of the reader, we put the proof here.

Lemma 2.1 [6]. *Consider Eq. (2.1) with the continuous functions $f = (f_1, \ldots f_n)$ satisfying:*

(**H**) *for all $t \geqslant 0$ and $\varphi \in C_n$ such that $|\varphi(\theta)| < |\varphi(0)|$ for $\theta \in [-\tau, 0)$, then $\varphi_i(0) f_i(t, \varphi) < 0$ for some $i \in \{1, \ldots, n\}$ such that $|\varphi(0)| = |\varphi_i(0)|$.*

Then, all solutions of (2.1) are defined and bounded for $t \geqslant 0$. Moreover, if $x(t) = x(t, 0, \varphi)$, with $\varphi \in C_n$, is a solution of (2.1), then $|x(t)| \leqslant \|\varphi\|$ for all $t \geqslant 0$.