Applied Mathematics and Computation 219 (2013) 9363–9371

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Improving approximate inverses based on Frobenius norm minimization ☆

Luis González *, Antonio Suárez

Department of Mathematics, University of Las Palmas de Gran Canaria, 35017 Las Palmas de Gran Canaria, Spain

ARTICLE INFO

Keywords:
Approximate inverses
Frobenius norm minimization
Trace
Spectrum
Singular values
Normal equations

ABSTRACT

Approximate inverses, based on Frobenius norm minimization, of real nonsingular matrices are analyzed from a purely theoretical point of view. In this context, this paper provides several sufficient conditions, that assure us the possibility of improving (in the sense of the Frobenius norm) some given approximate inverses. Moreover, the optimal approximate inverses of matrix $A \in \mathbb{R}^{n\times n}$, among all matrices belonging to certain subspaces of $\mathbb{R}^{n\times n}$, are obtained. Particularly, a natural generalization of the classical normal equations of the system $Ax = b$ is given, when searching for approximate inverses $N \neq A^T$ such that AN is symmetric and $\|AN - I\|_F < \left\|AA^T - I\right\|_F$.

1. Introduction

Let $\mathbb{R}^{n\times n}$ be the set of all $n \times n$ real matrices. In the following, $A \in \mathbb{R}^{n\times n}$ is assumed to be nonsingular, the symbols A^T, A^{-1} and $tr(A)$ stand for the transpose, the inverse, and the trace of matrix A, respectively, and I denotes the $n \times n$ identity matrix.

Roughly speaking, by a left (right, respectively) approximate inverse of A, we mean a matrix $N \in \mathbb{R}^{n\times n}$ such that the matrix product NA (AN, respectively) is "close to the identity" in a certain sense. This closeness may be measured by using an adequate matrix norm, and throughout this paper we use the Frobenius norm $\|\cdot\|_F$. Moreover, we address only the case of the right approximate inverses and, for simplicity, they will be referred to as approximate inverses (but analogous results can be obtained for the left ones). More precisely, we begin with the following definition.

Definition 1.1. Let $A, N, N' \in \mathbb{R}^{n\times n}$. Assume that A is nonsingular. Then we say that N is better approximate inverse of A than N', or that N improves N' as approximate inverse of A if and only if

$$\|AN - I\|_F < \left\|AN' - I\right\|_F.$$

In this context, given a linear subspace S of $\mathbb{R}^{n\times n}$, we consider the problem of obtaining the optimal approximate inverse N of matrix A in the subspace S. In accordance with Definition 1.1, throughout this paper the terms "the optimal" or "the best", mean that matrix $N \in S$ minimizes the Frobenius norm on the residual matrix $AM - I$. More precisely, "the optimal" approximate inverse N of A in S is the solution to the minimization problem

$$\min_{M\in S}\|AM - I\|_F = \|AN - I\|_F, \tag{1.1}$$

but the approximate inverse N is not necessarily optimal in any other sense of the word.

☆ Short running title: Improving approximate inverses

* Corresponding author.
E-mail address: luisglez@dma.ulpgc.es (L. González).

0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.03.057

The optimization problem (1.1) is tightly connected to the numerical analysis of linear systems

$$Ax = b, \quad A \in \mathbb{R}^{n\times n}, \quad x, b \in \mathbb{R}^n, \tag{1.2}$$

where A is a large, sparse and nonsingular matrix.

Indeed, in numerical linear algebra, the resolution of these systems is usually performed by iterative methods based on Krylov subspaces [1,2]. In general, the convergence of such Krylov methods is not assured or may be too slow. To improve their behavior, a preconditioning matrix N is used to transform the system (1.2) into the following equivalent system,

$$ANy = b, \quad x = Ny, \tag{1.3}$$

the so-called right preconditioned system, which is performed in order to get a preconditioned matrix AN, as close as possible to the identity [3], and N is called a (right) approximate inverse preconditioner of system (1.2).

Often, the preconditioners are parametrized by prescribed sparsity patterns [4,5], but we consider here a more general case of linear parametrization where the approximate inverse N, defined by Eq. (1.1), belongs to an arbitrary matrix subspace S of $\mathbb{R}^{n\times n}$. In [6], the authors introduce the idea to use Frobenius norm minimization for preconditioning problems. Some of the methods for constructing sparse approximate inverse preconditioners that are best approximations in the Frobenius norm, can be found, for instance, in [7–15] and in the references therein.

It is important to highlight here that the purpose of this paper is to provide purely theoretical results about the optimal approximate inverses $N \in S \subseteq \mathbb{R}^{n\times n}$ given by Eq. (1.1), in the theoretical context of Definition 1.1. However, our purpose is not to propose new algorithms for the numerical problem of preconditioning linear systems. Only, in a few cases, our results are related to computational strategies using special approximate inverse preconditioners based on Frobenius minimization.

The main goal of this paper is to apply several spectral properties of the matrix product AN (N being the solution to problem (1.1)) to obtain sufficient conditions for the existence of approximate inverses improving (in the sense of Definition 1.1) some given approximate inverses. By the way, we obtain the optimal approximate inverses of matrix A, among all the matrices belonging to certain linear subspaces of $\mathbb{R}^{n\times n}$.

For this purpose, in Section 2, we recall some useful expressions for matrix N, and several spectral properties of matrix AN. Next, Section 3 is devoted to establish our new results: the above mentioned sufficient conditions for improving approximate inverses, as well as the optimal approximate inverses of matrix A in certain matrix subspaces of $\mathbb{R}^{n\times n}$. Finally, conclusions are presented in Section 4.

2. Some preliminaries

Now, we present some preliminary results required to make this paper self-contained. For more details about these previous results and for their proofs, we refer the reader to [16–18].

2.1. Expressions for matrix N

Taking advantage of the well-known fact that the matrix Frobenius norm derives from an inner product, the solution N to problem (1.1) can be directly obtained via orthogonal projections. Here, and in the following, orthogonality is with respect to the Frobenius inner product $\langle\cdot,\cdot\rangle_F$. More precisely, using the orthogonal projection theorem, the matrix product AN is the orthogonal projection of the identity onto the subspace AS, as stated by the following Lemma [16].

Lemma 2.1. *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and let S be a linear subspace of $\mathbb{R}^{n\times n}$. Then, the solution to problem (1.1) is characterized by*

$$tr(A^t ANM^t) = tr(AM), \quad \forall M \in S \tag{2.1}$$

and the minimum Frobenius norm is

$$\|AN - I\|_F^2 = n - tr(AN).$$

Moreover,

$$\|AN\|_F^2 = tr(AN).$$

Eq. (2.1) implicitly gives the solution N to problem (1.1). For the purposes of this paper, it suffices to recall here the following simple explicit formula. The basic idea simply consists of expressing the orthogonal projection AN of the identity matrix onto the subspace AS by its expansion with respect to an orthonormal basis of AS [16].

Lemma 2.2. *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. Let S be a linear subspace of $\mathbb{R}^{n\times n}$ of dimension d, and $\{M_1, \ldots, M_d\}$ a basis of S such that $\{AM_1, \ldots, AM_d\}$ is an orthogonal basis of AS. Then, the solution to problem (1.1) is*

$$N = \sum_{i=1}^{d} \frac{tr(AM_i)}{\|AM_i\|_F^2} M_i \tag{2.2}$$