Applied Mathematics and Computation 219 (2013) 9372–9376

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Generalized Cesàro difference sequence spaces of non-absolute type involving lacunary sequences

Mikail Et

Department of Mathematics, Firat University, 23119 Elazığ, Turkey

ARTICLE INFO

Keywords:
Difference sequence
Lacunary sequence
Statistical convergence

ABSTRACT

In this paper we introduce and examine some properties of the sequence spaces $C(\Delta_v^m, \theta, (p)), C[\Delta_v^m, \theta, (p)], C_\infty(\Delta_v^m, \theta, (p)), C_\infty[\Delta_v^m, \theta, (p)], N_\theta(\Delta_v^m, (p)), S_\theta(\Delta_v^m)$ and study various properties and inclusion relations of these spaces. We also show that the space $S_\theta(\Delta_v^m)$ may be represented as $N_\theta(\Delta_v^m, (p))$ space.

1. Introduction

Let w be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\|_\infty = \sup | x_k |$, where $k \in \mathbb{N} = \{1, 2, \ldots\}$, the set of positive integers. Also by bs, cs, ℓ_1 and ℓ_p; we denote the spaces of all bounded, convergent, absolutely summable and p-absolutely summable series, respectively.

Let $\theta = (k_r)$ be the sequence of positive integers such that $k_0 = 0, 0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Then θ is called a lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} will be denoted by q_r. Lacunary sequences have been studied in [4,7,13,15,18].

The notion of difference sequence spaces was introduced by Kızmaz [20] and the notion was generalized by Et and Çolak [10]. Later on Et and Esi [11] generalized these sequence spaces to the following sequence spaces. Let $v = (v_k)$ be any fixed sequence of nonzero complex numbers and let m be a non-negative integer. Then,

$$\Delta_v^m(X) = \{x = (x_k) : (\Delta_v^m x_k) \in X\}$$

for $X = \ell_\infty, c$ or c_0, where $m \in \mathbb{N}, \Delta_v^0 x = (v_k x_k), \Delta_v^m x = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$ and so $\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$. The sequence spaces $\Delta_v^m(X)$ are Banach spaces normed by

$$\|x\|_\Delta = \sum_{i=1}^{m} | v_i x_i | + \|\Delta_v^m x_k\|_\infty$$

for $X = \ell_\infty, c$ or c_0. Recently the difference sequence spaces have been studied in [1–3,5,8,9,19,26,27,29,31–33].

The Cesàro sequence spaces Ces_p and Ces_∞ have been introduced by Shiue [25]. Jagers [16] has determined the Köthe duals of the sequence space Ces_p $(1 < p < \infty)$. It can be shown that the inclusion $\ell_p \subset Ces_p$ is strict for $1 < p < \infty$. Later on the Cesàro sequence spaces X_p and X_∞ of non-absolute type are defined by Ng and Lee [21,22].

Let X be a sequence space. Then X is called.

(i) *Solid* (or *normal*), if $(\alpha_k x_k) \in X$ for all sequences (α_k) of scalars with $| \alpha_k | \leqslant 1$ for all $k \in \mathbb{N}$, whenever $(x_k) \in X$,
(ii) *Symmetric*, if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where π is a permutation of $\mathbb{N}$,

E-mail address: mikailet@yahoo.com

0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2013.03.039

(iii) *Sequence algebra* if $x \cdot y \in X$, whenever $x, y \in X$.

2. Main results

In this section we prove some results involving the sequence spaces $C(\Delta_v^m, \theta, (p)), C[\Delta_v^m, \theta, (p)], C_\infty(\Delta_v^m, \theta, (p)), C_\infty[\Delta_v^m, \theta, (p)]$ and $N_\theta(\Delta_v^m, (p))$.

Definition 2.1. Let $p = (p_r)$ be a sequence of strictly positive real numbers. We define the following sequence spaces:

$$C(\Delta_v^m, \theta, (p)) = \left\{ x = (x_k) : \sum_{r=1}^{\infty} \left| h_r^{-1} \sum_{k \in I_r} \Delta_v^m x_k \right|^{p_r} < \infty \right\},$$

$$C[\Delta_v^m, \theta, (p)] = \left\{ x = (x_k) : \sum_{r=1}^{\infty} \left(h_r^{-1} \sum_{k \in I_r} |\Delta_v^m x_k| \right)^{p_r} < \infty \right\},$$

$$C_\infty(\Delta_v^m, \theta, (p)) = \left\{ x = (x_k) : \sup_r \left| h_r^{-1} \sum_{k \in I_r} \Delta_v^m x_k \right|^{p_r} < \infty \right\},$$

$$C_\infty[\Delta_v^m, \theta, (p)] = \left\{ x = (x_k) : \sup_r h_r^{-1} \sum_{k \in I_r} |\Delta_v^m x_k|^{p_r} < \infty \right\},$$

$$N_\theta(\Delta_v^m, (p)) = \left\{ x = (x_k) : \lim_r h_r^{-1} \sum_{k \in I_r} |\Delta_v^m x_k - L|^{p_r} = 0 \right\}.$$

We get the following sequence spaces from the above sequence spaces giving particular values to θ, p, v and m.

(i) If $p_r = p$ for all $r \in \mathbb{N}$ we write $C(\Delta_v^m, \theta, p), C[\Delta_v^m, \theta, p], C_\infty(\Delta_v^m, \theta, p), C_\infty[\Delta_v^m, \theta, p]$ and $N_\theta(\Delta_v^m, p)$ instead of $C(\Delta_v^m, \theta, (p)), C[\Delta_v^m, \theta, (p)], C_\infty(\Delta_v^m, \theta, (p)), C_\infty[\Delta_v^m, \theta, (p)]$ and $N_\theta(\Delta_v^m, (p))$ respectively.
(ii) If $p_r = 1$ for all $r \in \mathbb{N}$ we write $C(\Delta_v^m, \theta), C[\Delta_v^m, \theta], C_\infty(\Delta_v^m, \theta), C_\infty[\Delta_v^m, \theta,]$ and $N_\theta(\Delta_v^m)$ instead of $C(\Delta_v^m, \theta, (p)), C[\Delta_v^m, \theta, (p)], C_\infty(\Delta_v^m, \theta, (p)), C_\infty[\Delta_v^m, \theta, (p)]$ and $N_\theta(\Delta_v^m, (p))$ respectively.
(iii) In the case $\theta = (2^r)$ and $p_r = 1$ for all $r \in \mathbb{N}$ we shall write $C(\Delta_v^m), C[\Delta_v^m], C_\infty(\Delta_v^m), C_\infty[\Delta_v^m]$ and $N_\theta(\Delta_v^m)$ instead of $C(\Delta_v^m, \theta, (p)), C[\Delta_v^m, \theta, (p)], C_\infty(\Delta_v^m, \theta, (p)), C_\infty[\Delta_v^m, \theta, (p)]$ and $N_\theta(\Delta_v^m, (p))$ respectively. If $x \in N_\theta(\Delta_v^m)$, we say that x is Δ_v^m-lacunary strongly summable to L. If we take $m = 0$ and $v = (1, 1, 1, \ldots)$ then we obtain the sequence space N_θ introduced and investigated by Freedman et al. [13]. In the case $\theta = (2^r)$ we write $|\sigma_1|(\Delta_v^m)$ instead of $N_\theta(\Delta_v^m)$. If $x \in |\sigma_1|(\Delta_v^m)$, we say that x is Δ_v^m-strongly Cesàro summable to L.

The above sequence spaces contain some unbounded sequences for $m \geqslant 1$, for example let $x = (k^m)$, then $x \in C_\infty[\Delta_v^m, \theta, (p)]$ but $x \notin \ell_\infty$.

The proof of the following two results are easy, so we state without proof.

Theorem 2.2. *Let the sequence (p_r) be bounded. Then the sequence spaces $C(\Delta_v^m, \theta, (p)), C[\Delta_v^m, \theta, (p)], C_\infty(\Delta_v^m, \theta, (p)), C_\infty[\Delta_v^m, \theta, (p)]$ and $N_\theta(\Delta_v^m, (p))$ are linear spaces.*

Theorem 2.3. *Let m denote an arbitrary positive integer, then the following inclusions are strict.*

(i) $C(\Delta_v^{m-1}, \theta, p) \subset C(\Delta_v^m, \theta, p)$,
(ii) $C[\Delta_v^{m-1}, \theta, p] \subset C[\Delta_v^m, \theta, p]$,
(iii) $C[\Delta_v^m, \theta, (p)] \subset C(\Delta_v^m, \theta, (p))$,
(iv) $C(\Delta_v^m, \theta, p) \subset C(\Delta_v^m, \theta, q) (0 < p < q)$,

Theorem 2.4. *The sequence space $C[\Delta_v^m, \theta, p]$ is a BK-space normed by*

$$\|x\|_1 = \sum_{i=1}^{m} | v_i x_i | + \left(\sum_{r=1}^{\infty} \left(h_r^{-1} \sum_{k \in I_r} | \Delta_v^m x_k | \right)^p \right)^{\frac{1}{p}}, \quad (1 \leqslant p < \infty). \tag{1}$$

$C_\infty[\Delta_v^m, \theta]$ and $N_\theta(\Delta_v^m)$ *are BK-space normed by*