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# On the stabilization of Timoshenko systems with memory and different speeds of wave propagation

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ABSTRACT

In this work we consider a one-dimensional Timoshenko system with different speeds of wave propagation and with only one control given by a viscoelastic term on the angular rotation equation. For a wide class of relaxation functions and for sufficiently regular initial data, we establish a general decay result for the energy of solution. Unlike the past history and internal feedback cases, the second energy is not necessarily decreasing. To overcome this difficulty, a precise estimate of the second energy, in terms of the initial data and the relaxation function, is proved.



## 1. Introduction

In the present work we are concerned with the asymptotic behavior of the solution of the following Timoshenko system:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x+\psi)_x = 0 & \text{in} \quad ]0,L[\times\mathbb{R}_+, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi) + \int_0^t g(t-s)\psi_{xx}(s)ds = 0 & \text{in} \quad ]0,L[\times\mathbb{R}_+, \\ \varphi(0,t)=\psi(0,t)=\varphi(L,t)=\psi(L,t)=0 & \text{in} \quad \mathbb{R}_+, \\ \varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x) & \text{on} \quad ]0,L[, \\ \psi(x,0)=\psi_0(x), \quad \psi_t(x,0)=\psi_1(x) & \text{on} \quad ]0,L[, \end{cases} \tag{P}$$

where $t$ denotes the time variable, $x$ is the space variable along the beam of length $L$, in its equilibrium configuration, $\varphi$ is the transverse displacement of the beam, $\psi$ is the rotation angle of the filament of the beam, $g:\mathbb{R}_+\to\mathbb{R}_+$ is a non-increasing function, and the coefficients $\rho_1,\rho_2,k_1$ and $k_2$ are positive constants denoting, respectively, the density (the mass per unit length), the polar moment of inertia of a cross section, the shear modulus and Young's modulus of elasticity times the moment of inertia of a cross section and satisfying

$$\frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2}. \tag{1.1}$$

Our aim is to establish a general decay result, depending on $g$, for the energy of the system ($P$).

The Timoshenko system which describes the transverse vibration of a beam was first introduced in [24] and has the form

$$\begin{cases} \rho_1\varphi_{tt} = k_1(\varphi_x-\psi)_x & \text{in} \quad ]0,L[\times\mathbb{R}_+, \\ \rho_2\psi_{tt} = k_2\psi_{xx} + k_1(\varphi_x-\psi) & \text{in} \quad ]0,L[\times\mathbb{R}_+. \end{cases} \tag{1.2}$$

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Since then many people have been interested in the question of stability of (1.2) with different kind of controls: internal, boundary feedback, memory or past history. Let us mention some of these results.

If both the rotation angle and the transverse displacement are controlled, then it is well known that (1.2) is stable for any weak solution and without any restriction on the constants $\rho_1, \rho_2, k_1$ and $k_2$. Many decay estimates were obtained in this case; see for example [3,7–10,12,16,21,22,25–27].

If only the rotation angle is controlled, then there are two different cases. The case of different wave speed of propagation (1.1) and the opposite case. For the case $\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}$, it is well known that, similarly to the case of two controls, (1.2) is stable and similar decay results were obtained. We quote in this regard [2,4,5,11,13–15,17–20,23]. If (1.1) holds (which is more interesting from the physics point of view), then it is well known that (1.2) is not exponentially stable even for exponentially decaying relaxation functions. Moreover, some polynomial decay estimates for the strong solution of (1.2) were established only for the case of internal feedback in [1] and the case of past history in [15,20]. In these papers, the idea of the proof of the polynomial decay results exploits the non-increasingness property of the second energy (the energy of the system resulting from differentiating the original system with respect to time) to estimate some higher-order terms.

In the case of memory control $(P)$, the second energy is not necessarily non-increasing. To overcome this difficulty, we give an explicit estimate for the second energy in terms of the relaxation function and the initial data. In addition, we consider here a wider class of relaxation functions $g$ than those considered in the case of past history control [15,20].

The paper is organized as follows. In Section 2, we state some hypotheses and present our stability result. In Section 3, we give the proof of our stability result.

## 2. Preliminaries

We consider the following hypothesis:(H) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function satisfying

$$g(0) > 0, \quad k_2 - \int_0^{+\infty} g(s)ds =: l > 0 \tag{2.1}$$

and there exists a non-increasing differentiable function $\xi$: $\mathbb{R}_+ \to ]0, +\infty[$ and a constant $p \geqslant 1$ such that

$$g'(t) \leqslant -\xi(t)g^p(t), \quad \forall t \geqslant 0. \tag{2.2}$$

**Remark 2.1.** Condition (2.2) describes better the growth of $g$ at infinity and allows us to obtain precise estimate of the energy and more general than the "stronger" one ($\xi =$ constant and $p \in [1, \frac{3}{2}[$) used in the case of past history control [15,20]. We consider here the form (2.2) because our decay estimate can be expressed in a better way in the case $\xi =$ constant, than in the one $p = 1$.

**Remark 2.2.** By using a standard Galerkin method, we can show that $(P)$ has, for any initial data

$$(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in \left(H^2(]0, L[) \cap H_0^1(]0, L[)\right) \times H_0^1(]0, L[),$$

a unique (strong) solution

$$\varphi, \psi \in C\left(\mathbb{R}^+; H^2(]0, L[) \cap H_0^1(]0, L[)\right), \tag{2.3}$$

$$\cap C^1\left(\mathbb{R}^+; H_0^1(]0, L[)\right) \cap C^2\left(\mathbb{R}^+; L^2(]0, L[)\right),$$

and for any initial data

$$(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(]0, L[) \times L^2(]0, L[),$$

problem $(P)$ has a unique (weak) solution

$$\varphi, \psi \in C\left(\mathbb{R}^+; H_0^1(]0, L[)\right) \cap C^1\left(\mathbb{R}^+; L^2(]0, L[)\right). \tag{2.4}$$

Now we introduce the energy functional associated with $(P)$ by

$$E(t) := \frac{1}{2} g \circ \psi_x + \frac{1}{2}\int_0^L \left[\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \left(k_2 - \int_0^t g(s)ds\right)\psi_x^2 + k_1(\varphi_x + \psi)^2\right]dx, \tag{2.5}$$

where, for all $v : \mathbb{R}_+ \to L^2(]0, L[)$,

$$g \circ v = \int_0^L \int_0^t g(t-s)(v(t) - v(s))^2 ds dx. \tag{2.6}$$

Our main stability result reads: