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Nonconforming quadrilateral finite element method for a class of nonlinear sine–Gordon equations ☆

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ABSTRACT

Nonconforming quadrilateral finite element method (FEM) of the two-dimensional nonlinear sine–Gordon equation is studied for semi-discrete and Crank–Nicolson fully-discrete schemes, respectively. Firstly, we prove a special feature of a new arbitrary quadrilateral element (named modified Quasi–Wilson element), i.e., the consistency error is of order $O(h^2)$ (h denotes the mesh size) in H^1-norm, which leads to optimal order error estimate and superclose result with order $O(h^2)$ for the semi-discrete scheme through a different approach from the existing literature. Secondly, because the consistency error estimate of the new modified Quasi–Wilson element can reach a staggering $O(h^3)$ order, two orders higher than that of interpolation error, the optimal order error estimates of Crank–Nicolson fully-discrete scheme are obtained on arbitrary quadrilateral meshes with Ritz projection. Moreover, a superclose result in H^1-norm is presented on generalized rectangular meshes by a new technique. Thirdly, the global superconvergence results of H^1-norm for both semi-discrete and fully-discrete schemes are derived on rectangular meshes with interpolated postprocessing technique. Finally, a numerical test is carried out to verify the theoretical analysis.

1. Introduction

The nonlinear sine–Gordon equation arises in various problems in science and engineering. The study of this model mainly on two aspects: some attention has been paid to exact solution by algebraic analysis [1–5] and the others are devoted to numerical simulation with variety of numerical methods [6–16], such as finite difference methods, FEMs, pseudo spectral, domain decomposition methods and so on. It is well known that FEM is an important numerical method and has been widely used in evolution equations [17–20]. There are also some excellent work on FE approximation to two-dimensional nonlinear sine–Gordon equation [11–14]. However, to our best knowledge, almost all of previous analysis only concentrated on conforming FEs.

As we know, due to the consistency error, whether for linear or nonlinear problems, the use of nonconforming FEs may be unable to get the optimal order convergence estimates. So it is critical to select appropriate nonconforming FEs. Recently, one of the authors analyzed a class of low-order nonconforming elements for sine–Gordon equation, such as EQ_1^{rot}element [21,22], Q_1^{rot}element [23], P_1-nonconforming element [24], the optimal order error estimate order with $O(h)$ is obtained

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on rectangular meshes [25]. In this paper, we propose a new arbitrary quadrilateral element (named modified Quasi–Wilson element) and further explore its approximation properties to a class of nonlinear sine–Gordon equations.

On one hand, approaches used in [25] for error estimates are no longer valid to the new modified Quasi–Wilson element space V_h since it does not satisfy $\int_l [v_h]ds = 0$, where $[v_h]$ stands for the jump of $v_h \in V_h$ across the internal edge l. Thus we need to develop some new techniques to deal with this issue. Fortunately, with the similar argument to [26] for Quasi–Wilson element [27,28], it is shown that the new modified Quasi–Wilson element has a typical feature, i.e., the consistency error estimate in H^1-norm is of order $O(h^2)$, one order higher than its interpolation error. Then combining with the high accuracy result of bilinear FE on generalized rectangular meshes [29], we can derive the optimal order error estimate in L^2-norm and superclose result in H^1-norm with order $O(h^2)$ for the semi-discrete scheme, which improves the results of [25] by one order.

On the other hand, we find surprisely that when the exact solution belongs to $H^4(\Omega) \cap H_0^1(\Omega)$, the consistency error estimate of the new modified Quasi–Wilson element can reach a staggering $O(h^3)$ order, two order higher than its interpolation error. Then using the high accuracy result of bilinear FE and Ritz projection, and employing a new technique different from the above analysis of semi-discrete scheme, we get the optimal error estimates on arbitrary quadrilateral meshes and superclose property in H^1-norm with order $O(h^2 + (\triangle t)^2)$ ($\triangle t$ is the time step) on generalized rectangular meshes for Crank–Nicolson fully-discrete scheme, respectively. These results are one order higher than that of the usually low order nonconforming FE methods mentioned in [25]. Furthermore, applying interpolation postprocessing technique, we present the global superconvergence in H^1-norm for semi-discrete and Crank–Nicolson fully-discrete schemes with order $O(h^2)$ and $O(h^2 + (\triangle t)^2)$ respectively for the first time. We point out that the results provided herein are also valid to Quasi–Wilson on rectangular meshes (see Remark 3). Finally, numerical results are presented to confirm our theoretical analysis.

It is worth notice that some very popular nonconforming quadrilateral and triangular elements cannot be applied to sine–Gordon equation to get the results of this paper directly, such as the quasi-conforming isoparametric element [30], Wilson element, Carey element [31], Crouzeix–Raviart linear triangular element [32] and so on. In addition, how to use the nonconforming elements of [21–25] to get $O(h^2)$ order estimates still remains open.

The remainder of this paper is organized as follows: In next section, we present two important lemmas, then give the optimal error estimate and superclose result of semi-discrete scheme on generalized rectangular meshes. In Section 3, the optimal error estimates are obtained for Crank–Nicolson fully-discrete scheme on arbitrary quadrilateral meshes. Moreover, the superclose property in H^1-norm with order $O(h^2 + (\triangle t)^2)$ is derived on generalized rectangular meshes. Finally, we get the global superconvergence in H^1-norm for semi-discrete and Crank–Nicolson fully-discrete schemes on rectangular meshes. At the same time, we also carry out a numerical experiment to verify the performance of the new element.

We will use standard notations for the Sobolev spaces $H^m(\Omega)$ with norm $\|\cdot\|_m$ and semi-norm $|\cdot|_m$, and $H^m(K)$ with norm $\|\cdot\|_{m,K}$ and semi-norm $|\cdot|_{m,K}$, where $m \geqslant 0$ is an integer. Let $\|\cdot\|_0$ and $\|\cdot\|_{0,K}$ be the $L^2(\Omega)$-norm and $L^2(K)$-norm, respectively. Besides, let $P_k(K)$ be the space consisting of piecewise polynomials of degree k, and $Q_k(K)$ be space of polynomials whose degrees for x, y are equal to k on element K, where $k \geqslant 0$ is an integer. Throughout the paper, C denotes a positive constant independent of the mesh parameter h and may be different at each appearance.

2. Semi-discrete scheme

Consider the following sine–Gordon equation [1]

$$\begin{cases} u_{tt} + \alpha u_t - \gamma \triangle u + \beta \sin u = f(X,t), & (X,t) \in \Omega \times (0,T], \\ u(X,0) = u_0(X), & X \in \Omega, \\ \frac{\partial u}{\partial t}(X,0) = \varphi_0(X), & X \in \Omega, \\ u(X,t) = 0, & (X,t) \in \partial\Omega \times (0,T], \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded convex polygon domain with Lipschitz boundary $\partial\Omega, \alpha, \gamma$ and β are positive constants, $X = (x,y),\ f(X,t),\ u_0(X),\ \varphi_0(X)$ are given smooth functions.

Let $W = u_t$, then problem (1) is equivalent to

$$\begin{cases} \frac{\partial W}{\partial t} + \alpha W - \gamma \triangle u + \beta \sin u = f(X,t), & (X,t) \in \Omega \times (0,T], \\ \frac{\partial u}{\partial t} = W, & (X,t) \in \Omega \times [0,T], \\ u(X,0) = u_0(X), & X \in \Omega, \\ W(X,0) = \varphi_0(X), & X \in \Omega, \\ u(X,t) = 0, & (X,t) \in \partial\Omega \times (0,T]. \end{cases} \tag{2}$$

The variational form of problem (2) is to find $u, W \in H_0^1(\Omega)$, such that for all $v \in H_0^1(\Omega)$

$$\begin{cases} \left(\frac{\partial W}{\partial t}, v\right) + (\alpha W, v) + a(u,v) + (\beta \sin u, v) = (f, v), \\ \left(\frac{\partial u}{\partial t}, v\right) = (W, v), \\ (u(0) - u_0, v) = 0, \\ (W(0) - \varphi_0, v) = 0, \end{cases} \tag{3}$$

where $u(0) = u(X,0),\ W(0) = W(X,0),\ a(u,v) = \gamma \int_\Omega \nabla u \nabla v dX$.