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# 2-Iterated Appell polynomials and related numbers ☆

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ABSTRACT

This article deals with the introduction of Appell type sets of polynomials and associated numbers. The 2-iterated Appell polynomials are introduced and the Bernoulli and Euler based Appell polynomials are deduced as their particular cases. Different sets of polynomials, namely 2-iterated Bernoulli and Euler, Bernoulli–Euler (or Euler–Bernoulli) and their related numbers are considered. The operational relations between these families and Appell polynomials are used on the results of the Bernoulli and Euler polynomials to obtain the results for the corresponding mixed polynomials.



## 1. Introduction and preliminaries

The class of Appell sequences [1] is an important class of polynomial sequences and appears in different applications in pure and applied mathematics. The Appell sequences are characterized by Roman [16] in several ways. The Appell polynomials [1] may be defined by either of the following equivalent conditions:

$\{A_n(x)\}$ $(n \in \mathbb{N}_0)$ is an Appell set ($A_n$ being of degree exactly $n$), if either,

(i) $\frac{d}{dx}A_n(x) = nA_{n-1}(x) \quad (n \in \mathbb{N}_0)$, or
(ii) there exists an exponential generating function of the form

$$A(t)e^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \tag{1.1}$$

where $A(t)$ has (at least the formal) expansion:

$$A(t) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!} \quad (A_0 \neq 0). \tag{1.2}$$

Alternatively, the sequence $A_n(x)$ [16] is Appell for $g(t)$, if and only if

$$\frac{1}{g(t)}e^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \tag{1.3}$$

where

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!} \quad (g_0 \neq 0). \tag{1.4}$$

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In view of Eqs. (1.1) and (1.3), we have

$$A(t)=\frac{1}{g(t)}. \tag{1.5}$$

Next, we recall that according to the monomiality principle [17,5], a polynomial set $\{p_n(x)\}_{n\in\mathbb{N}}$ is "quasi-monomial", provided there exist two operators $\hat{M}$ and $\hat{P}$ playing, respectively, the role of multiplicative and derivative operators for the family of polynomials. These operators satisfy the following identities, for all $n\in\mathbb{N}$:

$$\hat{M}\{p_n(x)\}=p_{n+1}(x) \tag{1.6}$$

and

$$\hat{P}\{p_n(x)\}=np_{n-1}(x). \tag{1.7}$$

The operators $\widehat{M}$ and $\widehat{P}$ also satisfy the commutation relation

$$[\hat{P},\hat{M}]=\hat{P}\hat{M}-\hat{M}\hat{P}=\hat{1} \tag{1.8}$$

and thus display the Weyl group structure.

If $\widehat{M}$ and $\widehat{P}$ have differential realizations, then in view of monomiality principle [5] and combining Eqs. (1.6) and (1.7), we get the following differential equation satisfied by $p_n(x)$:

$$\widehat{M}\widehat{P}\{p_n(x)\}=np_n(x). \tag{1.9}$$

Assuming here and in the sequel $p_0(x)=1$, then the series definition for $p_n(x)$ can be obtained as:

$$p_n(x)=\widehat{M}^n\{1\}. \tag{1.10}$$

Also, identity (1.10) implies that the exponential generating function of $p_n(x)$ can be given in the form:

$$e^{\widehat{M}t}\{1\}=\sum_{n=0}^{\infty}p_n(x)\frac{t^n}{n!}\quad(|t|<\infty). \tag{1.11}$$

We note that the Appell polynomials $A_n(x)$ are quasi-monomial [17,14,5] with respect to the following multiplicative and derivative operators:

$$\widehat{M}_A=x+\frac{A'(D_x)}{A(D_x)}, \tag{1.12a}$$

or, equivalently

$$\widehat{M}_A=x-\frac{g'(D_x)}{g(D_x)} \tag{1.12b}$$

and

$$\widehat{P}_A=D_x, \tag{1.13}$$

respectively.

The class of Appell sequences contains a large number of classical polynomial sequences such as the Bernoulli, Euler, Hermite and Laguerre polynomials *etc*. The Bernoulli polynomials play an important role in various expansions and approximation formulae, which are useful both in analytic theory of numbers and in classical and numerical analysis. Jacob Bernoulli discovered the Bernoulli polynomials $B_n(x)$ to evaluate the sum $1^k+2^k+3^k+\cdots+(n-1)^k$ as $k!\int_0^n B_k(x)dx$. These polynomials appear in various problems in the fields of engineering and physics providing their solutions. We present the generating functions, series definitions and other properties of the Bernoulli and Euler polynomials and related numbers in Table 1.1.

**Table 1.1**
Results for the Bernoulli and Euler polynomials and related numbers.

| S. No. | Results | Bernoulli polynomials $B_n(x)$ | Bernoulli numbers $B_n$ | Euler polynomials $E_n(x)$ | Euler numbers $E_n$ |
|---|---|---|---|---|---|
| I. | $A(t)$; $g(t)$ | $\frac{t}{e^t-1};\frac{e^t-1}{t}$ | | $\frac{2}{e^t+1};\frac{e^t+1}{2}$ | |
| II. | Generating functions | $\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}$ [15, p. 299] | $\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}$ where $B_n:=B_n(0)$ [15] | $\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}$ [15, p. 300] | $\frac{2e^t}{e^{2t}+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}$ where $E_n:=2^nE_n(\frac{1}{2})$ [15] |
| III. | Multiplicative and derivative operators | $\hat{M}_B=x-\frac{(D_x-1)e^{D_x}+1}{D_x(e^{D_x}-1)}$, $\hat{P}_B=D_x$ | | $\hat{M}_E=x-\frac{e^{D_x}}{(e^{D_x}+1)}$, $\hat{P}_E=D_x$ | |
| IV. | Differential equations | $\left(xD_x-\frac{(D_x-1)e^{D_x}+1}{(e^{D_x}-1)}-n\right)\times B_n(x)=0$ | | $\left(xD_x-\frac{e^{D_x}D_x}{(e^{D_x}+1)}-n\right)\times E_n(x)=0$ | |
| V. | Series definitions | $B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}$ [15] | | $E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\times(x-\frac{1}{2})^{n-k}$ [15] | |