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On the range of Stockwell transforms

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ABSTRACT

The structure of the range of Stockwell transforms for a signal from $L_2(\mathbb{R})$ is obtained using the Vasilevski scheme. The paper completes the full picture of structural results for the four commonly used transforms of signal analysis: continuous wavelet transform, short-time Fourier transform, continuous shearlet transform and the Stockwell transform. Applications of this new characterization are discussed in connection with time–frequency signal localization.

1. Introduction and preliminaries

It is well-known that the short-time Fourier transform cannot track the signal dynamics properly for non-stationary signals due to the limitations of fixed window width, whereas the second time–frequency analysis tool, the continuous wavelet transform, is sensitive to noise. Thus, Robert Stockwell et al. proposed in [18] a method which has a frequency dependent resolution of time–frequency domain and entirely refer to local phase information. It can be seen as a multi-scale local Fourier transform providing a multi-resolution time–frequency data representation. For a signal (we identify a signal with an element $f \in L_2(\mathbb{R})$) and a window $\varphi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = 1$$

the *Stockwell transform* $S_\varphi f$ of f with respect to φ is defined by

$$(S_\varphi f)(b,\xi) = \frac{|\xi|}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\overline{\varphi(\xi(x-b))}\,\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}x$$

for all $b \in \mathbb{R}$ and $\xi \in \mathbb{R}\setminus\{0\}$. Putting the Stockwell transform in perspective, we note that for all $f \in L_2(\mathbb{R})$, $b \in \mathbb{R}$ and $\xi \in \mathbb{R}\setminus\{0\}$ it holds

$$(S_\varphi f)(b,\xi) = \langle f, \varphi^{b,\xi}\rangle, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $L_2(\mathbb{R})$,

$$\varphi^{b,\xi} = \frac{1}{\sqrt{2\pi}} M_\xi T_{-b} D_\xi \varphi$$

and M_ξ, T_{-b} and D_ξ are the modulation, translation and dilation operators given by

$$(M_\xi h)(x) = \mathrm{e}^{\mathrm{i}x\xi} h(x),$$

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$$(T_{-b}h)(x) = h(x-b),$$

$$(D_\xi h)(x) = |\xi|\, h(\xi x)$$

for all measurable functions h on $\mathbb{R}$ and all $x \in \mathbb{R}$, respectively.

A fundamental connection between the Stockwell S_φ and the continuous wavelet transform W_ψ is clear from the definition, see [18], i.e., for all $f \in L_2(\mathbb{R})$ holds

$$(S_\varphi f)(b,\xi) = \sqrt{\frac{|\xi|}{2\pi}}\, e^{-ib\xi}(W_\psi f)\left(b, \frac{1}{\xi}\right), \quad b \in \mathbb{R},\ \xi \in \mathbb{R} \setminus \{0\},$$

where

$$\psi(x) = e^{ix}\varphi(x), \quad x \in \mathbb{R}.$$

Further details may be found in [8,19]. From this observation it is easy to obtain the following resolution of identity for the Stockwell transform, see [6,8]. Let

$$\mathcal{F}\{g\}(\omega) := \hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x)\, e^{-ix\omega}\, dx$$

be the Fourier transform on $L_2(\mathbb{R})$ (angular-frequency convention).

Theorem 1.1. *Let $\varphi \in L_2(\mathbb{R})$ be a normalized window such that*

$$\int_{\mathbb{R}\setminus\{0\}} |\hat{\varphi}(\xi-1)|^2 \frac{d\xi}{|\xi|} = 1. \tag{2}$$

Then for all $f, g \in L_2(\mathbb{R})$ it holds $\langle f, g\rangle = (S_\varphi f, S_\varphi g)$, where $(\cdot,\cdot)$ is the inner product in the Hilbert space $L_2\left(\mathbb{R}\times\mathbb{R}\setminus\{0\}, \frac{db\,d\xi}{|\xi|}\right)$.

This result implies that for all $f \in L_2(\mathbb{R})$ the Stockwell transform $S_\varphi f$ is an element of $L_2\left(\mathbb{R}\times\mathbb{R}\setminus\{0\}, \frac{db\,d\xi}{|\xi|}\right)$ and

$$\|S_\varphi f\|_{L_2(\mathbb{R}\times\mathbb{R}\setminus\{0\}, \frac{db\,d\xi}{|\xi|})} = \|f\|_{L_2(\mathbb{R})},$$

which means that the operator $S_\varphi : L_2(\mathbb{R}) \to L_2\left(\mathbb{R}\times\mathbb{R}\setminus\{0\}, \frac{db\,d\xi}{|\xi|}\right)$ given by (1) is an isometry. Also it is easy to see that the range

$$S_\varphi(L_2(\mathbb{R})) = \{S_\varphi f; f \in L_2(\mathbb{R})\}$$

of the Stockwell transform is a closed subspace of $L_2\left(\mathbb{R}\times\mathbb{R}\setminus\{0\}, \frac{db\,d\xi}{|\xi|}\right)$. Moreover, Theorem 1.1 implies that for all $F \in S_\varphi(L_2(\mathbb{R}))$ we have

$$\begin{aligned} F(b,\xi) &= (S_\varphi f)(b,\xi) = \langle f, \varphi^{b,\xi}\rangle \\ &= \int_{\mathbb{R}\times\mathbb{R}\setminus\{0\}} (S_\varphi f)(b',\xi')\overline{(S_\varphi \varphi^{b,\xi})(b',\xi')}\, \frac{db'd\xi'}{|\xi'|} \\ &= \int_{\mathbb{R}\times\mathbb{R}\setminus\{0\}} K(b,\xi;b',\xi')F(b',\xi')\, \frac{db'd\xi'}{|\xi'|}, \end{aligned}$$

where

$$K(b,\xi;b',\xi') = \overline{(S_\varphi \varphi^{b,\xi})(b',\xi')} = \langle \varphi^{b',\xi'}, \varphi^{b,\xi}\rangle$$

for all $(b,\xi), (b',\xi') \in \mathbb{R}\times\mathbb{R}\setminus\{0\}$. This means that $S_\varphi(L_2(\mathbb{R}))$ is a reproducing kernel Hilbert space. Now, the integral operator

$$(P_\varphi F)(b,\xi) = \int_{\mathbb{R}\times\mathbb{R}\setminus\{0\}} F(b',\xi')K(b,\xi;b',\xi')\, \frac{db'd\xi'}{|\xi'|}$$

is the orthogonal projection onto $S_\varphi(L_2(\mathbb{R}))$.

Nowadays the Stockwell transform has gained popularity in the signal analysis community because of its simplicity and usefulness to study applied problems, covering areas as geophysics, oceanology, engineering, optics, bioinformatics or biomedicine. See the recent papers [5,16,17,21] for a few diverse applications of the Stockwell transform. Some mathematical aspects of the transform are also of recent interests and they are under development in different directions. For instance, the underlying group structure is described in [2], its modification in [9], and the associated localization operators are investigated in [14,15]. In this note we are interested in the range $S_\varphi(L_2(\mathbb{R}))$ of Stockwell transform. We follow the general scheme of Vasilevski presented in [20] which was already successfully used in the case of continuous wavelet transform [10] as well as short-time Fourier transform [13]. See also [1], where the author studies the structure of Gabor and super Gabor spaces, in particular those which are generated by vectors of Hermite functions. Applying this scheme for our particular case we char-