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A blow-up criterion for compressible nematic liquid crystal flows

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ABSTRACT

In this paper, we prove a regularity criterion for the local strong solutions to a simplified hydrodynamic flow modeling the compressible, nematic liquid crystal materials in a bounded domain.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$, we consider the following simplified version of Ericksen-Leslie system modeling the hydrodynamic flow of compressible nematic liquid crystals:

$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla p(\rho) - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div}\, u = -\Delta d \cdot \nabla d, \tag{1.2}$$

$$\partial_t d + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, |d| = 1 \text{ in } \Omega \times (0,\infty), \tag{1.3}$$

$$u = 0, \quad d = d_0(x) \text{ on } \partial\Omega \times (0,\infty), \tag{1.4}$$

$$(\rho, u, d)(x,0) = (\rho_0, u_0, d_0)(x), \quad |d_0| = 1, \quad x \in \Omega \subseteq \mathbb{R}^3. \tag{1.5}$$

Here ρ is the density of the fluid, u is the fluid velocity and d represents the macroscopic average of the nematic liquid crystal orientation field, $p(\rho) := a\rho^\gamma$ is the pressure with positive constants $a > 0$ and $\gamma \geqslant 1$. μ and λ are the shear viscosity and the bulk viscosity coefficients of the fluid respectively, which are assumed to satisfy the following physical condition:

$$\mu > 0, \quad 3\lambda + 2\mu \geqslant 0.$$

(1.1) and (1.2) is the well-known compressible Navier–Stokes system with the external force $-\Delta d \cdot \nabla d$. (1.3) is the well-known heat flow of harmonic map when $u = 0$.

Very recently, Huang et.al. [1] proved the following local-in-time well-posedness.

Proposition 1.1. *Let $\rho_0 \in W^{1,q}$ for some $q \in (3,6]$ and $\rho_0 \geqslant 0$ in $\Omega, u_0 \in H^2, d_0 \in H^3$ and $|d_0| = 1$ in Ω. If, in additions, the following compatibility condition*

$$-\mu\Delta u_0 - (\lambda+\mu)\nabla div\, u_0 - \nabla p(\rho_0) - \Delta d_0 \cdot \nabla d_0 = \sqrt{\rho_0} g \text{ for some } g \in L^2(\Omega) \tag{1.6}$$

holds, then there exist $T_0 > 0$ and a unique strong solution (ρ, u, d) to the problem (1.1)–(1.5).

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Based on the above Proposition, Huang et. al. [2] proved the following regularity criterion

$$\int_0^T \|\mathcal{D}(u)\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2 dx < \infty,$$

to the problem (1.1)–(1.3),(1.5) with the boundary condition

$$u = 0 = \frac{\partial d}{\partial v} \text{ on } \partial\Omega \times (0, \infty).$$

or

$$u \cdot v = \operatorname{curl} u \times v = \frac{\partial d}{\partial v} = 0 \text{ on } \partial\Omega \times (0, \infty).$$

Here

$$\mathcal{D}(u) := \frac{1}{2}(\nabla u + {}^t\nabla u).$$

v is the unit outward normal vector to $\partial\Omega$.

This result generalizes that in [3,4]. However, the methods in [2] can not be used here directly. When the term $|\nabla d|^2 d$ in (1.3) is replaced by $\frac{1}{4}(1 - |d|^2)d$, the problem (1.1)–(1.5) has been studied by Liu and Liu [5], they proved the following regularity criterion

$$\int_0^T \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^\infty} dt < \infty.$$

The aim of this paper is to study the regularity criterion of local strong solutions to the problem (1.1)–(1.5). We will prove

Theorem 1.2. *Let the assumptions in Proposition 1.1 hold true. If*

$$\int_0^T \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2 dt < \infty, \tag{1.7}$$

then the solution can be extended beyond $T > 0$.

Remark 1.1. When the boundary condition $u = 0$ is replaced by

$$u \cdot v = 0, \quad \operatorname{curl} u \times v = 0,$$

the same result holds true. For recent results on the nematic liquid crystal flows, we refer to [6].

2. Proof of Theorem 1.2

Since (ρ, u, d) is the local strong solution, we only need to prove the a priori estimates.

First, by the same calculations as that in [2], it is easy to show that

$$\|\rho\|_{L^\infty(0,T;L^\infty)} \leqslant C, \tag{2.1}$$

$$\int \rho|u|^2 + |\nabla d|^2 dx + \int_0^T \int |\nabla u|^2 + |\Delta d|^2 dx dt \leqslant C. \tag{2.2}$$

By the Gagliardo–Nirenberg inequality

$$\|u\|_{L^\infty}^2 \leqslant C\|u\|_{L^6}^{\frac{4}{3}} \|\nabla u\|_{L^\infty}^{\frac{2}{3}} \leqslant C\|\nabla u\|_{L^2}^{\frac{4}{3}} \|\nabla u\|_{L^\infty}^{\frac{2}{3}} \leqslant C\|\nabla u\|_{L^\infty} + C\|\nabla u\|_{L^2}^4,$$

we see that

$$\int_0^T \|u\|_{L^\infty}^2 dt \leqslant C. \tag{2.3}$$

By the Hölder inequality

$$\|\nabla u\|_{L^3}^2 \leqslant C\|\nabla u\|_{L^2}^{\frac{4}{3}} \|\nabla u\|_{L^\infty}^{\frac{2}{3}} \leqslant C\|\nabla u\|_{L^\infty} + C\|\nabla u\|_{L^2}^4,$$

we find that

$$\int_0^T \|\nabla u\|_{L^3}^2 dt \leqslant C. \tag{2.4}$$