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Solvability of second-order Hamiltonian systems with impulses via variational method

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ABSTRACT

In this paper, a class of second-order impulsive Hamiltonian systems are considered. Some new existence results are obtained by using a variational method and critical point theorem due to Tang and Wu. Some recent results are extended. Three examples are presented to illustrate the feasibility and effectiveness of our results.

1. Introduction

In this paper, we will investigate the existence of solution for the following non-autonomous second-order Hamiltonian systems with impulsive effects

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \\ \Delta(\dot{u}^i(t_j)) = I_{ij}(u^i(t_j)), \quad i = 1, 2, \ldots, N, \ j = 1, 2, \ldots, l, \end{cases} \tag{1}$$

where $u(t) = (u^1(t), u^2(t), \ldots, u^N(t)), t_0 = 0 < t_1 < t_2 < \cdots < t_l < t_{l+1} = T, T > 0, \Delta(\dot{u}^i(t_j)) = \dot{u}^i(t_j^+) - \dot{u}^i(t_j^-)$, where $\dot{u}^i(t_j^+)$ and $\dot{u}^i(t_j^-)$ denote the right and left limits of $\dot{u}^i(t)$ at $t = t_j$, respectively, impulsive functions $I_{ij} : \mathbb{R} \to \mathbb{R}(i = 1, 2, \ldots, N, j = 1, 2, \ldots, l)$ are continuous, $\nabla F(t, x)$ is the gradient of $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ with respect to x and F satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+), b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| \leqslant a(|x|)b(t), \quad |\nabla F(t, x)| \leqslant a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

For the sake of convenience, in the sequel, we define $\mathcal{A} = \{1, 2, \ldots, N\}, \mathcal{B} = \{1, 2, \ldots, l\}$. $(\cdot, \cdot)$ and $|\cdot|$ denote the usual inner product and usual norm in $\mathbb{R}^N$, respectively.

Let $F(t, x) = -K(t, x)$ and $I_{ij} \equiv 0$ for all $i \in \mathcal{A}, j \in \mathcal{B}$, then (1) is Hamiltonian system

$$\begin{cases} -\ddot{u}(t) = \nabla K(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{2}$$

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Many existence results are obtained for problem (2) by critical point theory, such as [1–5] and their references. Using the reduction method, least action principle and minimax methods, some solvability conditions for (2) are obtained by Zhao and Wu [6–8]. Very recently, by the reduction method, the perturbation argument and the least action principle, Tang and Wu [9] got a critical point theorem without the compactness assumptions. And by using the critical point theorem, they also obtained some existence results for (2), among which Theorem A and Theorem B (see below) unify and generalize some corresponding results of [6–8]. For the reader's convenience, we now recall three results for (2) in [9].

Theorem A [9, Theorem 1.3]. *Suppose that assumption (A) holds and there exists $\mu \in L^1(0,T;\mathbb{R})$ with $\int_0^T \mu(t)dt > 0$ such that $K(t,x) - \mu(t)|x|^2/2$ is convex in x for a.e. $t \in [0,T]$. Assume that*

(A1) There exist $\alpha \in L^1(0,T;\mathbb{R}^+)$ with $\int_0^T \alpha(t)dt < 12/T$ and $\gamma \in L^1(0,T;\mathbb{R}^+)$ such that $K(t,x) \leqslant \alpha(t)|x|^2/2 + \gamma(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Then problem (2) has at least one solution in H_T^1.

Theorem B [9, Theorem 1.4]. *Suppose that assumption (A) holds and there exists $k \in L^1(0,T;\mathbb{R}^+)$ with $\int_0^T k(t)dt < 12/T$ such that $-K(t,x) + k(t)|x|^2/2$ is convex in x for a.e. $t \in [0,T]$. Assume that*

(A2) $\int_0^T K(t,x)dt \to +\infty$ as $|x| \to \infty, x \in \mathbb{R}^N$.

Then problem (2) has at least one solution in H_T^1.

Theorem C [9, Theorem 3.3]. *Suppose that assumption (A) holds and $K(t,x)$ is convex in x for a.e. $t \in [0,T]$. Assume that (A1) and (A2) hold. Then problem (2) has at least one solution in H_T^1.*

On the other hand, impulsive effects exist widely in many evolution processes in which their states are changed abruptly at certain moments of time. The theory of impulsive differential systems has been developed by numerous mathematicians (see e.g. [10–15]). Applications of impulsive problems occur in control theory, biology, population dynamics, chemotherapeutic treatment in medicine and so on (see e.g. [16–24]).

Furthermore, there have been many approaches to study impulsive problems, such as method of upper and lower solutions with the monotone iterative technique, fixed point theory and topological degree theory. Recently, variational method was employed to study the existence and multiplicity of solutions for impulsive problems (see e.g. [14,15,25–31]).

More precisely, Zhou and Li [14] obtained some sufficient conditions for the existence of solution for (1). Sun et al. [15] got the multiplicity of solutions for the following impulsive Hamiltonian systems with a perturbed term

$$\begin{cases} -\ddot{u} + A(t)u = \lambda \nabla F(t,u) + \mu \nabla G(t,u), \quad \text{a.e. } t \in [0,T], \\ \Delta(\dot{u}^i(t_j)) = I_{ij}(u^i(t_j)), \quad i = 1,2,\ldots,N,\ j = 1,2,\ldots,l, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases}$$

In order to obtain the existence and multiplicity of solutions, impulsive functions I_{ij} of all theorems in [14,15] are required to satisfy

$$I_{ij}(y)y \geqslant 0 \quad \text{for any } i \in \mathcal{A}, j \in \mathcal{B} \text{ and } y \in \mathbb{R}, \tag{3}$$

or

$$I_{ij}(y)y \leqslant 0 \quad \text{for any } i \in \mathcal{A}, j \in \mathcal{B} \text{ and } y \in \mathbb{R}. \tag{4}$$

However, there are many functions which are not possible to satisfy (3) or (4). For example, when $N = 3$ and $l = 2$, impulsive functions of (1) are

$$I_{ij}(y) = -y + 1 \quad \text{for all } i = 1,2,3,\ j = 1,2, \tag{5}$$

or more complicated case

$$I_{ij}(y) = \begin{cases} y/2 + 1, & i = 1,2,3,\ j = 1, \\ -y, & i = 1,2,\ j = 2, \\ y/2, & i = 3,\ j = 2. \end{cases} \tag{6}$$

Remark 1. When the impulsive functions of (1) are (5) or (6), examples are considered by the results of this paper in Section 3.

Motivated by Tang and Wu [9] and the above facts, the aim of this paper is to revisit problem (1) and study the existence of solution without assumption (3) or (4). The main role of (3) in [14] is to ensure the existence of a bounded minimizing sequence of the corresponding functional. Condition (4) in [14] plays an important role in ensuring the boundedness of