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Common fixed point of power contraction mappings satisfying (E.A) property in generalized metric spaces

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ABSTRACT

In this paper, using the setting of generalized metric spaces, existence of unique common fixed point of two pairs of power contraction mappings satisfying (E.A)-property in generalized metric spaces is established. We also provide example to support the results presented herein.

1. Introduction and preliminaries

Mustafa and Sims [21] generalized the concept of a metric in which the real number is assigned to every triplet of an arbitrary set. Based on the notion of generalized metric spaces, Mustafa et al. [22–26] obtained some fixed point theorems for mappings satisfying different contractive conditions. Chugh et al. [15] obtained some fixed point results for maps satisfying property p in $G-$ metric spaces. Saadati et al. [29] studied fixed point of contractive mappings in partially ordered $G-$ metric spaces. Shatanawi [32] obtained fixed points of $\Phi-$ maps in $G-$ metric spaces. Radenović et al. [28] proved some tripled coincidence point results in $G-$ metric spaces. Recently, Nashine et al. [27] obtained coincidence and fixed point results in ordered $G-$ metric spaces. The study of unique common fixed points of mappings satisfying strict contractive conditions has been at the center of rigorous research activity. Abbas and Rhoades [2] initiated the study of common fixed point theorems in generalized metric spaces (see also, [3,5,6]). Further results in the direction of common fixed points in generalized metric space are obtained by [4,19]. Also, see [8–11,31]. The aim of this paper is to study common fixed point of weakly compatible power contraction maps satisfying (E.A)-property in the framework of $G-$ metric spaces.

Consistent with Mustafa and Sims [22], the following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to 211d^{+}$ satisfies:

- $\mathbf{G}_1$: $G(x,y,z) = 0$ if $x = y = z$;
- $\mathbf{G}_2$: $0 < G(x,y,z)$ for all $x,y,z \in X$, with $x \neq y$;
- $\mathbf{G}_3$: $G(x,x,y) \leqslant G(x,y,z)$ for all $x,y,z \in X$, with $y \neq z$;
- $\mathbf{G}_4$: $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$, (symmetry in all three variables); and
- $\mathbf{G}_5$: $G(x,y,z) \leqslant G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$.

Then G is called a $G-$ metric on X and (X,G) is called a $G-$metric space.

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Definition 1.2. A sequence $\{x_n\}$ in a $G-$ metric space X is:

(i) a $G-$ *Cauchy* sequence if, for any $\varepsilon > 0$, there is an $n_0 \in N$ (the set of natural numbers) such that for all $n, m, l \geqslant n_0, G(x_n, x_m, x_l) < \varepsilon$,
(ii) a $G-$ *convergent* sequence if, for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in N$, such that for all $n, m \geqslant n_0, G(x, x_n, x_m) < \varepsilon$.

A $G-$ metric space on X is said to be $G-$ complete if every $G-$Cauchy sequence in X is $G-$ convergent in X. It is known that $\{x_n\}$ $G-$ converges to $x \in X$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$.

Proposition 1.3. *Let X be a G− metric space. Then the following are equivalent*:

(1) $\{x_n\}$ *is G− convergent to x*.
(2) $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$.
(3) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$.
(4) $G(x_n, x, x) \to 0$ *as* $n \to \infty$.

Definition 1.4. A $G-$ metric on X is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 1.5. *Every G− metric on X induces a metric d_G on X given as follows*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \text{for all} \quad x, y \in X. \tag{1.1}$$

For a symmetric G− metric

$$d_G(x, y) = 2G(x, y, y), \quad \text{for all} \quad x, y \in X. \tag{1.2}$$

However, if G is non-symmetric, then the following inequality holds:

$$\frac{3}{2}G(x, y, y) \leqslant d_G(x, y) \leqslant 3G(x, y, y), \quad \text{for all} \quad x, y \in X. \tag{1.3}$$

It is also obvious that

$$G(x, x, y) \leqslant 2G(x, y, y).$$

We not that subset A in G-metric space is bounded if it is bounded in metric d_G.

Now, we give an example of a non-symmetric G− metric.

Example 1.6. Let $X = \{1, 2\}$ and a mapping $G : X \times X \times X \to 211d^+$ be defined as:

(x,y,z)	$G(x,y,z)$
$(1,1,1), (2,2,2)$	0
$(1,1,2), (1,2,1), (2,1,1)$	0.5
$(1,2,2), (2,1,2), (2,2,1)$	1.

Note that G satisfies all the axioms of a generalized metric but $G(x, x, y) \neq G(x, y, y)$ for distinct x, y in X. Therefore G, is a non-symmetric $G-$ metric on X.

Sessa [30] introduced the notion of the weak commutativity of mappings in metric spaces. Recently Abbas et al. [6] studied $R-$ weakly commuting and compatible mappings in the frame work of $G-$ metric spaces.

Definition 1.7 [6]. Let X be a $G-$ metric space. Mappings $f, g : X \to X$ are called (i) weakly commuting if $G(fgx, fgx, gfx) \leqslant G(fx, fx, gx)$, for all $x \in X$ (ii) $R-$ weakly commuting if there exists a positive real number R such that $G(fgx, fgx, gfx) \leqslant RG(fx, fx, gx)$ holds for each $x \in X$ (iii) compatible if, whenever a sequence $\{x_n\}$ in X is such that $\{fx_n\}$ and $\{gx_n\}$ are $G-$ convergent to some $t \in X$, then $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n) = 0$ (iv) noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $\{fx_n\}$ and $\{gx_n\}$ are $G-$ convergent to some $t \in X$, but $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n)$ is either nonzero or does not exist.

Self mappings f and g on X are said to be weakly compatible if $fx = gx$ implies $fgx = gfx$ ([18]). Thus f and g are weakly compatible if and only if f and g are pointwise R-weakly commuting mappings.

In 2002, Aamri and Moutaawakil [1] introduced (E.A) property to obtain common fixed point of two mappings. Recently, Babu and Negash [12] employed this concept to obtain some new common fixed point results (see also [13,14,17]).