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One-switch utility functions with annuity payments

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ABSTRACT

This paper derives the functional forms of multiattribute utility functions that lead to a maximum of one-switch change in preferences between any two uncertain and multi-period cash flows as the decision maker's wealth increases through constant annuity payments. We derive the general and continuous non-constant solutions of the corresponding functional equations.

1. Introduction

One of the most important steps in decision analysis is determining the decisions maker's utility function [12]. Several authors have discussed this issue and have presented methods to assess and derive the functional form of a single-attribute utility function based on its risk aversion properties [6,10], or by the change in valuation of a lottery as the decision maker's wealth increases [1,2,7,9]. In particular, Pfanzagl [9] showed that if the decision maker's preferences between any two uncertain and uni-period lotteries does not change as the decision maker's initial wealth changes, then he must have either a linear or an exponential utility function. Pfanzagl characterized such utility functions by the functional equation

$$W(x+z) = k(z)W(x) + \ell(z).$$

Bell [7] further developed this notion and introduced the idea of characterizing a utility function based on the maximum number of switches that may occur between any two lotteries as the decision maker's wealth increases. To illustrate, suppose that a decision maker prefers lottery A to lottery B. Now suppose that all outcomes of the lotteries are modified by a shift amount z. If the decision maker's preference between the lotteries does not change for any value of z, then he must have either a linear or an exponential utility function. Thus linear and exponential utility functions are 0-switch utility functions. On the other hand, if preferences between the two lotteries can change, but can change only once, as we increase z, then the decision maker is said to have a 1-switch utility function. The extension to m-switch utility functions is straightforward; there m is the maximum number of preference changes that can occur as we increase z. Bell [7] characterized the functional forms of m-switch utility functions. Abbas and Bell [4] (see also [2]) showed that a one-switch utility function, U, must satisfy the system of functional equations

$$U(x+z) = K(z)U(x) + M(z)W(x) + L(z),$$
$$W(x+z) = k(z)W(x) + \ell(z).$$

In many cases that arise in practice, a decision maker may face multi-period and uncertain cash flows. Abbas, Aczél, and Chudziak [3] discussed the functional forms of multiattribute utility functions that lead to zero-switch change in preferences between multi-period cash flows when a decision maker's initial wealth increases through an annuity that pays a constant

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amount z every time period. This paper derives the functional forms of multiple attribute utility functions that lead to a maximum of one-switch change in preferences. In particular, we consider one-switch preferences over uncertain n-period cash flows as the decision maker's initial wealth increases. The initial wealth is in the form of an annuity payment that pays an equal amount, z, every period for n successive periods, and we consider the solutions of the following system of functional equations:

$$U(x_1+z,\dots,x_n+z)=K(z)U(x_1,\dots,x_n)+M(z)W(x_1,\dots,x_n)+L(z), \tag{1}$$

$$W(x_1+z,\dots,x_n+z)=k(z)W(x_1,\dots,x_n)+\ell(z). \tag{2}$$

The remainder of this paper is structured as follows: Section 2 presents the problem formulation and notation. Section 3 presents several Lemmas and preliminary results. Section 4 presents the main results and the general and continuous non-constant solutions to the system (1) and (2).

2. Problem formulation

Assume that D is a non-empty open subset of $\mathbb{R}^n$ ($n \geqslant 2$),

$$V_{(x_1,\dots,x_n)} := \{z\in\mathbb{R} \mid (x_1+z,\dots,x_n+z)\in D\} \quad \text{for } (x_1,\dots,x_n)\in D,$$

$$V_D := \bigcup_{(x_1,\dots,x_n)\in D} V_{(x_1,\dots,x_n)},$$

$$T := \{(x_2-x_1,\dots,x_n-x_1) \mid (x_1,\dots,x_n)\in D\}$$

and, for every $(t_1,\dots,t_{n-1})\in T$,

$$V^{(t_1,\dots,t_{n-1})} := \bigcup_{(x_1,\dots,x_n)\in D,(x_2-x_1,\dots,x_n-x_1)=(t_1,\dots,t_{n-1})} V_{(x_1,\dots,x_n)}.$$

Furthermore, given a function $\psi: T \to \mathbb{R}$, we set

$$V_{\psi\neq 0} := \bigcup_{(x_1,\dots,x_n)\in D,\psi(x_2-x_1,\dots,x_n-x_1)\neq 0} V_{(x_1,\dots,x_n)}.$$

Let us recall that a function $a:\mathbb{R}\to\mathbb{R}$ is said to be *additive*, provided it satisfies $a(x+y)=a(x)+a(y)$ for $x,y\in\mathbb{R}$; and a function $e:\mathbb{R}\to\mathbb{R}$ is said to be *exponential*, provided $e(x+y)=e(x)e(y)$ for $x,y\in\mathbb{R}$. It is well known (see e.g. [5]) that every additive function $a:\mathbb{R}\to\mathbb{R}$ continuous at a point has the form $a(z)=az$ for $z\in\mathbb{R}$ with some real constant a. Moreover, every non-zero exponential function $e:\mathbb{R}\to\mathbb{R}$ continuous at a point has the form $e(z)=e^{\alpha z}$ for $z\in\mathbb{R}$ with some real constant α. In particular, every non-constant additive or exponential function is non-constant on every interval.

We consider the system of functional Eqs. (1) and (2) for $(x_1,\dots,x_n)\in D$ and $z\in V_{(x_1,\dots,x_n)}$, where $U,W:D\to\mathbb{R}$ and $K,L,M,k,\ell : V_D\to\mathbb{R}$ are unknown functions. Eq. (2) has been already solved in [3] under the assumptions that D is open, $V_{(x_1,\dots,x_n)}$ is an interval for every $(x_1,\dots x_n)\in D$ and a function

$$V_{(x_1,\dots,x_n)} \ni z \to W(x_1+z,\dots,x_n+z) \tag{3}$$

is non-constant for atleast one $(x_1,\dots,x_n)\in D$. It is not difficult to check that in fact [3, Theorem 4.3] remains true (with the same proof) if, instead of the openness of D, we assume that, for every $(x_1,\dots,x_n)\in D$, the set $V_{(x_1,\dots,x_n)}$ is an *open* interval. Let us recall that result in such a modified version.

Theorem 2.1. *Let D be a nonempty subset of $\mathbb{R}^n$ such that $V_{(x_1,\dots,x_n)}$ is an open interval for every $(x_1,\dots,x_n)\in D$. Assume that $W:D\to\mathbb{R}, k,\ell : V_D\to\mathbb{R}$ and a function given by (3) is non-constant for atleast one $(x_1,\dots,x_n)\in D$. Then a triple (W,k,ℓ) satisfies* Eq. (2) *if and only if one of the following two conditions holds.*

(s1) *There exist a non-constant additive function $a:\mathbb{R}\to\mathbb{R}$ and a function $\psi : T\to\mathbb{R}$ such that*

$$\begin{cases} k(z)=1 & \text{for} \quad z\in V_D,\\ \ell(z)=a(z) & \text{for} \quad z\in V_D,\\ W(x_1,\dots,x_n)=\psi(x_2-x_1,\dots,x_n-x_1)+a(x_1) & \text{for} \quad (x_1,\dots,x_n)\in D.\end{cases}$$

(s2) *There exist a non-constant exponential function $e:\mathbb{R}\to\mathbb{R}$, a constant $c\in\mathbb{R}$ and a not identically zero function $\psi : T\to\mathbb{R}$ such that*

$$\begin{cases} k(z)=e(z) & \text{for} \quad z\in V_{\psi\neq 0},\\ \ell(z)=c(1-k(z)) & \text{for} \quad z\in V_D,\\ W(x_1,\dots,x_n)=e(x_1)\psi(x_2-x_1,\dots,x_n-x_1)+c & \text{for} \quad (x_1,\dots,x_n)\in D.\end{cases}$$